Local Fields in Boundary Conformal QFT

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Dedicated to Detlev Buchholz on the occasion of his 60th birthday

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Abstract

Conformal field theory on the half-space $x > 0$ of Minkowski spacetime ("boundary CFT") is analyzed from an algebraic point of view, clarifying in particular the algebraic structure of local algebras and the bi-localized charge structure of local fields. The field content and the admissible boundary conditions are characterized in terms of a non-local chiral field algebra.

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1 Introduction

We study relativistic boundary CFT on the half-plane $M_+ = \{(t,x) : x > 0\}$. This is a local QFT with a conserved and traceless stress-energy tensor, subject to a boundary condition at the boundary $x = 0$. As is well known, conservation and vanishing of the trace imply that the components $T_L = \frac{1}{2}(T_{00} + T_{01})$ and $T_R = \frac{1}{2}(T_{00} - T_{01})$ are chiral fields, $T_L = T_L(t+x)$, $T_R = T_R(t-x)$. The boundary condition is the absence of energy flow across the boundary,

$$T_{01}(t, x = 0) = 0 \qquad \Leftrightarrow \qquad T_L = T_R \equiv T. \tag{1.1}$$

It follows that the components $T_{10} = T_{01}$, $T_{11} = T_{00}$ of the stress-energy tensor are of the form

$$T_{00}(t,x) = T(t+x) + T(t-x), \qquad T_{01}(t,x) = T(t+x) - T(t-x), \tag{1.2}$$

i.e., *bi-local* expressions in terms of the chiral field T (cf. Fig. 1).

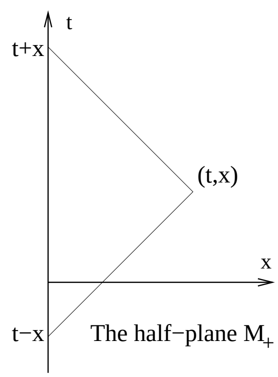

Fig. 1: A point in the half space M_+. A canonical field localized at (t,x) is a bi-local linear combination of chiral field localized at $t+x$ and $t-x$.

Apart from the stress-energy tensor, the theory may contain further chiral fields, such as currents, subject to an appropriate boundary condition; e.g., for a conserved current with $j_L = \frac{1}{2}(j_0 + j_1) = j_L(t+x)$, $j_R = \frac{1}{2}(j_0 - j_1) = j_L(t-x)$, the vanishing of the charge flow across the boundary gives

$$j_1(t, x = 0) = 0 \qquad \Leftrightarrow \qquad j_L = j_R \equiv j, \tag{1.3}$$

and

$$j_0(t,x) = j(t+x) + j(t-x), \qquad j_1(t,x) = j(t+x) - j(t-x). \tag{1.4}$$

It is crucial to contrast the bilocal form (1.2), (1.4) of the chiral fields in boundary CFT with the situation in 2D Minkowski space CFT, where, e.g.,

the stress-energy tensor has the chiral decomposition

$$T_{00}(t,x) = T(t+x)\otimes\mathbf{1}+\mathbf{1}\otimes T(t-x), \quad T_{01}(t,x) = T(t+x)\otimes\mathbf{1}-\mathbf{1}\otimes T(t-x), \tag{1.5}$$

where $T_L = T\otimes\mathbf{1}$ and $T_R = \mathbf{1}\otimes T$ are two *independent* (left and right) chiral fields. A boundary CFT contains only one chiral algebra with an appropriate identification between left and right movers. Consequently, the representation space is a direct sum of representations of the chiral algebra, rather than of tensor products of representations of two chiral algebras. This ought to be ascribed to the fact that the imposing of boundary conditions and the ensuing breakdown of symmetry have so drastic consequences on the ground state fluctuations (Casimir effect) that states respecting the boundary conditions cannot be realized in the Hilbert space of states without boundary conditions, see, e.g., [25].

Let us point out, however, that *locally* the two situations with $T_L = T_R = T$ and with $T_L = T\otimes\mathbf{1}$ and $T_R = \mathbf{1}\otimes T$ independent, are *algebraically indistinguishable*: for instance, in the latter case the commutator $[T_L(t_1,x_1)\pm T_R(t_1,x_1), T_L(t_2,x_2)\pm T_R(t_2,x_2)]$ involves only δ-function contributions at $t_1+x_1 = t_2+x_2$ and at $t_1-x_1 = t_2-x_2$, while the commutator $[T(t_1,x_1)\pm T(t_1,x_1), T(t_2,x_2)\pm T(t_2,x_2)]$ has additional contributions at $t_1+x_1 = t_2-x_2$ and at $t_1-x_1 = t_2+x_2$. But within a wedge region $M_+ \supset W : x > |t|$ ($\Leftrightarrow t-x < 0 < t+x$), the latter contributions are ineffective. The same holds for any time translate of W.

A slightly stronger version of this algebraic indistinguishability is the following: It has been shown that the chiral stress-energy tensor satisfies the *split property*: namely for every pair of intervals $J < I$ which do not touch (thus allowing to smooth out the UV singularities), there exists a state φ in the vacuum Hilbert space $\mathcal{H}_0$ of T (depending on I and J; in particular not the vacuum state) which has no correlations among $T(u_1)$ and $T(u_2)$ when $u_1 \in I$ and $u_2 \in J$. In other words, φ factorizes on products of $T(u_i)$ with $u_i \in I\cup J$ according to

$$\varphi\Big(\prod_k T(u_k)\Big) = \varphi\Big(\prod_{i:u_i\in I} T(u_i)\Big)\cdot\varphi\Big(\prod_{j:u_j\in J} T(u_j)\Big). \tag{1.6}$$

This implies that, for every double-cone O not touching the boundary (hence $t-x$ and $t+x$ belong to non-touching intervals as before), there is a state φ such that products of $T_{\mu\nu}(t,x)$ given by (1.2) in the boundary CFT with $(t,x)\in O$ have the same expectation values in the state φ as the same products of $T_{\mu\nu}(t,x)$ given by (1.5) in the 2D Minkowski space CFT have in the state $\varphi\otimes\varphi$. This property exhibits the local “decoupling” of left

and right chiral components. Exactly as the split property fails when the intervals I and J touch, the decoupling of left- and right-movers breaks down at the boundary in BCFT.

We shall assume the split property for *all* chiral fields of a boundary CFT. This property is known to be related to phase space properties of the CFT (existence of Tr $\exp -\beta L_0$) [9, 1], and it has been established for large classes of chiral models ([45] and references therein).

Let us now turn to local fields in boundary CFT which do not decompose in the manner of (1.2) or (1.4). These non-chiral fields have to satisfy local commutativity with the chiral fields and with each other, and transform covariantly under the conformal (Möbius) group generated by the chiral stress-energy tensor T. The starting point in the present article will be the crucial observation that

non-chiral local fields in BCFT arise from non-local chiral fields

by an algebraic construction (explained in detail in Sect. 2). This construction gives also rise to a model-independent explanation (Sect. 5) for an observation due to Cardy [11] concerning the structure of correlation functions. Cardy has shown that n-point functions of primary local fields in boundary CFT satisfy the same differential equations in the $2n$ variables $t_i \pm x_i$ as chiral $2n$-point conformal blocks of an associated two-dimensional Minkowski space CFT, and are therefore particular combinations of the latter. E.g., the 4-point function of the order parameter in the critical Ising model in the full plane factorizes as

$$\begin{aligned}\langle\Omega\,,\ \sigma(t_1,x_1)\,\sigma(t_2,x_2)\,\sigma(t_3,x_3)\,\sigma(t_4,x_4)\ \Omega\rangle &= \\ = F(t_1+x_1,\dots,t_4+x_4)\cdot F(t_1-x_1,\dots,t_4-x_4) &+ \\ +G(t_1+x_1,\dots,t_4+x_4)\cdot G(t_1-x_1,\dots,t_4-x_4) & \qquad (1.7)\end{aligned}$$

(where the chiral 4-point conformal blocks F and G correspond to intermediate states in the vacuum sector and in the "energy" sector, respectively), whereas both

$$\langle\Omega\,,\ \phi_0(t_1,x_1)\,\phi_0(t_2,x_2)\ \Omega\rangle \propto F(t_1+x_1,t_1-x_1,t_2+x_2,t_2-x_2) \qquad (1.8)$$

and

$$\langle\Omega\,,\ \phi_1(t_1,x_1)\,\phi_1(t_2,x_2)\ \Omega\rangle \propto G(t_1+x_1,t_1-x_1,t_2+x_2,t_2-x_2) \qquad (1.9)$$

are 2-point functions of local fields on the half-plane M_+. Expressed in terms of exchange fields [42] ("generalized chiral creation and annihilation

operators"), we have the operator factorization

$$\sigma(t,x) = a(t+x) \otimes a(t-x) + b(t+x) \otimes b(t-x) + \text{h.c.} \tag{1.10}$$

on the Hilbert space $[\mathcal{H}_0 \otimes \mathcal{H}_0] \oplus \left[\mathcal{H}_{\frac{1}{16}} \otimes \mathcal{H}_{\frac{1}{16}}\right] \oplus \left[\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}\right]$, where $\mathcal{H}_h$ are the three sectors of the stress-energy tensor with $c = \frac{1}{2}$, the exchange fields $a : \mathcal{H}_0 \to \mathcal{H}_{\frac{1}{16}}$ and $b : \mathcal{H}_{\frac{1}{16}} \to \mathcal{H}_{\frac{1}{2}}$ and their adjoints interpolate among the three sectors of the chiral stress-energy tensor, and $F = \langle a^*aa^*a\rangle$, $G = \langle a^*b^*ba\rangle$. In contrast, (1.8) and (1.9) are two-point functions of local fields on the half-plane, given by

$$\phi_0(t,x) \propto a^*(t+x)\, a(t-x) \tag{1.11}$$

defined on $\mathcal{H}_0$, and

$$\phi_1(t,x) \propto b(t+x)\, a(t-x) + a^*(t+x)\, b^*(t-x) \tag{1.12}$$

defined on $\mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}}$. The local commutativity at space-like distance of the combinations (1.11), (1.12) can be directly checked in terms of the exchange (braid group) commutation relations among a, b and their adjoints [42].[1] In this calculation, the specific ordering $t_1 - x_1 < t_2 - x_2 < t_2 + x_2 < t_1 + x_1$ (or $1 \leftrightarrow 2$) is crucial. In particular, the combinations ϕ_0 and ϕ_1 given by (1.11), (1.12) on the *entire* plane would fail to be *local* fields.

We learn from this explicit example that the local fields in boundary CFT carry a bi-localized product of charges of the chiral algebra, rather than a tensor product of left and right charges, as in Minkowski space CFT. Moreover, they interpolate in very specific ways among the charged sectors of the chiral algebra, and these structures determine the scaling behavior of the fields as $x \to 0$. E.g., the field ϕ_0 has a singular behavior $\propto x^{-\frac{2}{16}}$ as $x \to 0$, while the field ϕ_1 vanishes $\propto x^{\frac{1}{2}-\frac{2}{16}}$ at the boundary.[2] Thus, we also see that the choice of a boundary condition is related to the bi-localized charge structure of the local fields. We shall investigate the origin of this charge structure in the general case.

[1]More precisely, while any linear combination ϕ_1 of the two terms in (1.12) satisfies local commutativity with itself, it does so with ϕ_1^* only if ϕ_1 is a multiple of a hermitean field. ((1.12) *is* hermitean up to a phase due to the exchange comutation relations $b(t-x)a(t+x) = \omega b(t+x)a(t-x)$ and $a^*(t-x)b^*(t+x) = \omega a^*(t+x)b^*(t-x)$ with $\omega = \exp -i\frac{3}{8}\pi$ [42].) On the other hand, any two hermitean combinations differ only by a unitary Klein transformation; thus up to a global phase and unitary similarity, the combination (1.12) is unique as a local quantum field.

[2]In the general case, one argues as follows. As $x \to 0$, the variables $t+x$ and $t-x$ coalesce. Thus, the scaling bahvior is controlled by the operator product expansion, and depends on the particular fusion channel selected by the bi-localization formula.

For these purposes, we look at boundary CFT from the algebraic point of view [21]. The algebraic point of view emphasizes the representation theoretic features of a QFT, especially charges and their composition [12], rather than kinematical features such as analytic properties of correlation functions. The DHR theory of superselection sectors [12] asserts that all information about charges (superselection sectors), their composition ("fusion", operator product expansions), and their interchange ("statistics", commutation relations) is encoded in a braided C* tensor category (the DHR category for short), in terms of local observable quantities. This theory has been developed further into a powerful tool, useful for explicit computational purposes especially in the chiral setting.

E.g., the classification of local and non-local extensions of a given local QFT has been cast into a problem of classification of *Q-systems* ([33], see Sect. 4 and App. A) within the DHR category. Q-systems are an efficient tool to control the algebraic consistency of commutation relations, operator product expansions, and charge conjugation of primary and descendant fields at one stroke. Under the natural assumption of "complete rationality" (Sect. 2), the classification of irreducible chiral extensions in CFT has thus been shown to be a finite-dimensional problem with finitely many solutions (see Sect. 3.2). In the case $c < 1$, a complete classification has been obtained along these lines [28].

Furthermore, the existence of exchange fields as in (1.10)–(1.12) with numerical braid group commutation relations and their operator product expansion could be established from general principles in the algebraic approach [15, 16].

Rather than the local fields, say $\phi(t,x)$, the prime objects in the algebraic approach to QFT are the von Neumann algebras of local observables generated by the fields smeared with localized test functions, say

$$A(O) := \{\phi(f), \phi(f)^* : \operatorname{supp} f \subset O\}'' \tag{1.13}$$

for open space-time regions O. The properties of the assignment $O \mapsto A(O)$ (the *net of local algebras*) are axiomatized such that their generation by fields as in (1.13) becomes in fact obsolete and needs not be assumed at all.

In our case, the chiral fields $T(u)$, $(j(u), \dots)$ generate a chiral net of local von Neumann algebras

$$I \mapsto A(I), \qquad I = (a,b) \subset \mathbb{R} \tag{1.14}$$

on the vacuum Hilbert space $\mathcal{H}_0$. In fact, A extends to a net over the intervals of the circle (embedding $\mathbb{R}$ into S^1 by means of a Cayley transformation).

The chiral fields of a boundary CFT generate a net

$$O \mapsto A_+(O). \tag{1.15}$$

According to the prescription (1.13), $A_+(O)$ is generated by chiral fields smeared in the variable $t+x$ over the interval I and in the variable $t-x$ over the interval J, where $O = I \times J$, $I > J$, is an open double-cone in M_+. The bi-localized structure (1.2), (1.4) etc., translates into the form of the local algebras (cf. Fig. 2)

$$A_+(O) = A(I) \vee A(J) \qquad (O = I \times J, \;\; I > J). \tag{1.16}$$

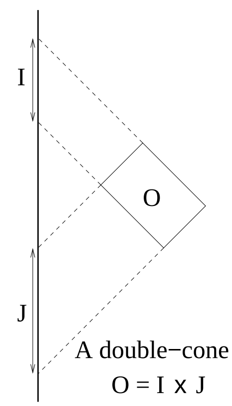

Fig. 2: A double-cone in the half space M_+. An observable in $A_+(O)$ is generated by chiral observables localized in I and J.

The (searched for) non-chiral local fields of the boundary CFT will generate a net of local algebras in their vacuum representation

$$O \mapsto B_+(O), \tag{1.17}$$

with $B_+(O)$ containing $A_+(O)$ and consequently commuting with $A_+(\hat{O})$ as $\hat{O}$ is space-like from O. But while A_+ is defined on the vacuum representation space $\mathcal{H}_0$ of the chiral CFT A, the boundary CFT B_+ will in general be defined on a larger Hilbert space $\mathcal{H}_B \supset \mathcal{H}_0$, i.e., one has

$$\pi(A_+(O)) \subset B_+(O). \tag{1.18}$$

Since we assume covariance and positive energy throughout, the representation π of A on $\mathcal{H}_B$ is a positive-energy representation, containing the vacuum representation, with irreducible decomposition $\pi \simeq \bigoplus_s n^s \cdot \pi_s$, $n^0 = 1$.

We take the structure (1.14)–(1.18) as the characteristic structure of algebraic boundary conformal QFT, irrespective whether the nets are generated as in (1.13) by any specific set of generating local fields. Our main

results will be the following (for any fixed chiral CFT A), referring to the body of the article for more detailed qualifications of the statements:

In Sect. 2, we provide several general results about the structure of local boundary CFT's B_+. Every maximal local boundary CFT B_+ can be recovered from its "restriction to the boundary". The latter is some (possibly non-local) chiral extension $I \mapsto B(I)$ of the chiral CFT $I \mapsto A(I)$ (Prop. 2.9), defined on the same Hilbert space $\mathcal{H}_B$ as B_+. Structural features of the latter (reviewed and developed in Sect. 3, where Tomita's Modular Theory [43] plays a crucial role) are exploited to infer structural features of boundary CFT. We shall refer to the (re)construction of the boundary CFT from a (non-local) chiral theory as *(boundary) induction*.

These results show that the classification of (non-local) chiral extensions of a local chiral theory (e.g., in terms of Q-systems) at the same time provides a classification of boundary CFT's.

On the other hand, every (non-local) chiral extension B of A determines a local CFT B_2^α on two-dimensional Minkowski space-time with left and right chiral observables $A \otimes A$ (henceforth referred to as the *α-induction construction*). The Hilbert space of B_2^α carries the representation $\pi_2 \simeq \bigoplus_{\sigma\tau} Z_{[\sigma][\tau]} \cdot \pi_\sigma \otimes \pi_{\bar\tau}$ of $A \otimes A$, where the matrix Z with indices in the set of irreducible sectors of A is a modular invariant determined by the chiral extension B [40, Cor. 1.6].

In Sect. 4, we discuss the relation between these two constructions of 2D nets (boundary induction for the half-space vs. α-induction for Minkowski space). Indeed, the local inclusions $\pi(A_+(O)) \subset B_+(O)$ and $\pi_2(A(I) \otimes A(J)) \subset B_2^\alpha(O)$ are algebraically isomorphic (Thm. 4.1). In this sense, the boundary CFT constitutes a representation of the local degrees of freedom of the Minkowski space theory B_2^α which is consistent with the chiral boundary condition (1.1) and its generalizations such as (1.3).

But the representation spaces of B_+ and of B_2^α are very different, one being a direct sum of sectors of A, the other being a direct sum of tensor products of sectors. Therefore inspite of the *algebraic* isomorphism, the bi-localized charge structure of the local fields on the half-space must be structurally different from the tensor product charge structure of the local fields in the plane, as is clearly exemplified by (1.11) or (1.12) vs. (1.10).

We derive an explicit formula for the local charged fields (Prop. 5.1) exhibiting their bi-localized charge structure in terms of non-local chiral exchange operators. The charge of a field $\phi(t,x)$ is a product (not a *tensor* product) of two chiral charges localized at $t+x$ and $t-x$, respectively. This structure, and as a consequence the behavior of the charged fields and their

correlations close to the boundary, is determined by the non-local chiral extension B, i.e., the choice of B "determines the boundary conditions".

In this sense, the natural reasoning where one would impose the boundary conditions first, and then attempt to construct local fields subject to these conditions, is inverted. This avoids the problem with the usual strategy, that a consistent set of boundary conditions must be chosen in the first place, while it is not a priori clear what "consistent" would mean. As our analysis demonstrates implicitly, the algebraic constraints on the local fields to be constructed are highly involved: they consist in (a) the Q-system describing the algebraic structure of the inclusion $A_+(O) \subset B_+(O)$, and (b) the representation of this algebraic structure on a Hilbert space $\mathcal{H}_B$. From these data which most sensitively depend on the DHR structure of the underlying chiral net A, the boundary conditions emerge, so that it is very unlikely that it should be possible to "guess" the consistent sets of boundary conditions without further specific insight. For this reason, we consider the present top-down strategy

$$\text{chiral extension} \rightarrow \text{boundary condition}$$

much more effective, since it is completely under control in the algebraic framework.

In Sect. 6 we show that along with a given (non-local) chiral extension B, there is a whole family of non-local chiral extensions B_a, all associated with the same Minkowski space theory B_2^α, and hence a family of boundary CFT nets $B_{a,+}$, which are all locally isomorphic, but whose local fields exhibit different bi-localized charge structures and satisfy different boundary conditions, in the sense just explained. The multiplicities of the Hilbert spaces $\mathcal{H}_a \equiv \mathcal{H}_{B_a} = \bigoplus_s n_a^s \cdot \mathcal{H}_s$ are the diagonal elements of a "nimrep" (non-negative integer matrix representation) of the fusion rules of A:

$$n^s \cdot n^t = \sum_u N_u^{st}\, n^u \qquad \text{with} \qquad n_{aa}^s = n_a^s. \tag{1.19}$$

We include in Sect. 7 some preliminary remarks on the relation to the modular structure of partition functions and boundary states.

The structural analysis pursued in this article generalizes closely related previous analyses in complementary approaches. In the context of critical phenomena in Statistical Mechanics, Cardy has already discussed [11] the case $B = A$ (in our terminology), leading to the set of boundary conditions being labelled by the sectors of A. The same situation was investigated by Felder, Fröhlich, Fuchs and Schweigert [14] from the perspective of three-dimensional topological field theory. Fuchs, Runkel and Schweigert [19]

proceeded to construct the coefficients of all $2n$-point conformal blocks as in (1.8), (1.9) in a combinatorial manner, where a condition very similar to our eq. (5.12) was crucial to ensure locality. Behrend *et al.* [10] have concentrated on graph theoretic aspects of the pertinent fusion algebras, and to A-D-E classification aspects in the case of $SU(2)$ current algebras, see also [46] for a review. Fuchs and Schweigert [17] have studied the generalization in which (in our terminology) A is a subtheory (not necessary of orbifold type) of a chiral theory B which is itself local. They also emphasized the role of α-induction. This case is known to give rise to block diagonal modular invariant matrices Z_{st} [6], and to this case also applies the result in [27]. The same authors [18] have further developed the purely categorial aspects characteristic of boundary CFT, no longer referring to the underlying physical postulates. In fact, these structures fit most naturally in the general setting of tensor categories as exposed, e.g., in [18, 30, 37].

In comparison to such a considerable gain of mathematical generality (where quantum physics remains hardly visible), the motivation and ambition of our work is more limited. On the other hand, we study and explain specific representation theoretic issues which in the other frameworks are not or even cannot be addressed. For these issues, operator algebraic methods are most powerful.

We emphasize that in our approach the prominent principle is Locality. In other approaches [46], inspired by Statistical Mechanics or String Theory rather than Quantum Field Theory, Modular Invariance of the partition function is taken instead as a first principle, required in order to guarantee that the theory can be consistently defined on arbitrary Riemann surfaces. It is well known, however that – although closely related to each other – these principles cannot be precisely mapped onto each other [41].

In fact, we do not assume diffeomorphism invariance but only Möbius invariance. Assuming diffeomorphism invariance (i.e., the algebraic implementation of localized diffeomorphisms by suitable chiral observables), would allow some stronger results. E.g., (for an explanation of the notions, see the beginning of the next section), it was shown in [35] that strong additivity would be automatic in a split net of finite μ-index, and that the μ-index coincides with the dimension of the DHR category. Concerning boundary CFT, one could infer that the index of the inclusion of the chiral observables in the BCFT observables associated with a double-cone, does not depend on the double-cone, in spite of the fact that the Möbius group does not act transitively on the double-cones in M_+.

2 Algebraic boundary conformal QFT

We work with a fixed chiral conformal net $I \mapsto A(I)$ over the intervals of the real axis [20], e.g., a Virasoro net with $c < 1$ or a non-abelian current algebra (affine Kac-Moody) chiral net. In this article, A is assumed to be *completely rational* [29]. This condition combines *rationality* (finitely many superselection sectors, each with finite statistics [12, 15]), *strong additivity* ("irrelevance of points for smearing", i.e., the algebras of two adjacent intervals (a, b) and (b, c) generate the algebra of the full interval (a, c); this property is equivalent to Haag duality of the chiral theory on the real line), and the *split property* (statistical independence of local algebras $A(I)$ and $A(J)$ when I and J are finitely separated, and as a consequence $A(I) \vee A(J)$ is isomorphic to $A(I) \otimes A(J)$, cf. the discussion around (1.6); this property is guaranteed, e.g., if $\exp -\beta L_0$ is a trace class operator for all β in the vacuum representation [9, 1]). Most of the common models of chiral CFT are completely rational [32, 45], but abelian current algebras as well as stress tensors with $c \geq 1$ without further fields are excluded by the assumption of rationality.

Completely rational chiral theories enjoy very interesting properties concerning the structure of their superselection sectors. E.g., the DHR statistics is non-degenerate (besides the vacuum sector, no sector has trivial monodromy with every other sector) [29, Cor. 37], and thus gives rise to a unitary representation of the modular group $SL(2, \mathbb{Z})$ in terms of the statistics [16, Cor. 5.2], turning the DHR category into a *modular category* [44]. Moreover, in completely rational theories, the dimension of the DHR category (the sum of the squares of the dimension of all irreducible superselection sectors), equals the "μ-index" (the von Neumann subfactor index of the inclusion $A(E) \subset A(E')'$ where E is the union of two disconnected intervals and E' its complement on the circle [29, Thm. 33]).

2.1. Geometric preliminaries on the half-space M_+.

Before turning to QFT on the half-space $M_+ \equiv \{(t, x) \in \mathbb{R}^2 : x > 0\}$, let us mention some elementary geometric properties of this space:

(i) A double-cone *within* the half-space M_+ is an open region of the form $O = I \times J \equiv \{(t, x) : t + x \in I,\ t - x \in J\}$ whose closure is contained in M_+ (cf. Fig. 3). Let $L \subset \mathbb{R}$ be a bounded open interval, and $J < K < I$ the three subintervals (ordered as indicated) obtained by removing two points from L (cf. Figs. 2 and 5). There is a bijection between the configurations of four intervals I, J, K, L obtained this way and the double-cones O within

M_+, such that $O = I \times J$. By default, I, J, K, L and O will always refer to such a configuration. Only if necessary, we shall write I_O, J_O, K_O, L_O in order to indicate this convenient parametrization of double-cones within M_+.

(ii) A left wedge is a region of the form $W_L = \{(t, x) : |t - t_0| < x_0 - x\}$ for some $(t_0, x_0) \in M_+$; it is "spanned" by the interval $I = (t_0 - x_0, t_0 + x_0)$. A right wedge is a region of the form $W_R = \{(t, x) : |t - t_0| < x - x_0\}$ for some $(t_0, x_0) \in M_+$. The causal complement of a left wedge is a right wedge, and vice versa (cf. Fig. 3). The causal complement of a double-cone is the union of a left wedge $W_L = O_<$ and a right wedge $W_R = O_>$. A double-cone $\hat{O}$ belongs to the left causal complement $O_<$ of O ($\hat{O} < O$) iff $L_{\hat{O}} \subset K_O$, and to the right causal complement $O_>$ ($\hat{O} > O$) iff $L_O \subset K_{\hat{O}}$.

Locality of a net B_+ on M_+ means that $B_+(\hat{O})$ commutes with $B_+(O)$ in both cases.

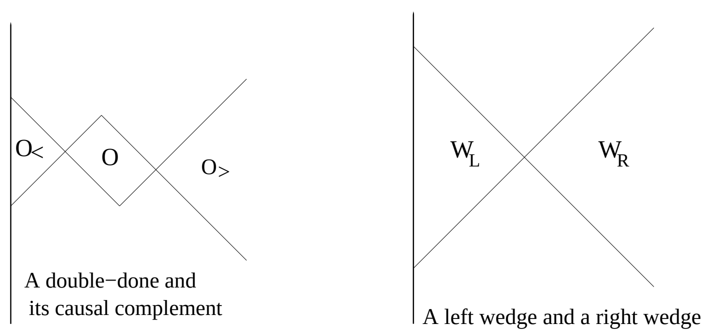

Fig. 3: Double-cone and wedge regions in M_+ and their causal complements.

(iii) The covering of the Möbius group $G := \widetilde{PSL}(2, \mathbb{R})$, acting on the universal covering of the compactification S^1 of $\mathbb{R}$, induces an action on a certain covering of $M_+ \subset \mathbb{R} \times \mathbb{R}$. The subgroups of translations and of dilations act on $\mathbb{R}$, and the induced actions are the time translations and the dilations of M_+, respectively.

2.2. Local algebras in boundary CFT.

2.1 Definition: A given chiral net A defines two different local nets over the open double-cones within M_+, namely the *trivial boundary CFT*

$$O \mapsto A_+(O) := A(I) \vee A(J) \tag{2.1}$$

and its *dual*

$$O \mapsto A_+^{\mathrm{dual}}(O) := A(L) \cap A(K)'. \tag{2.2}$$

As emphasized by the notation, $A_+^{\rm dual}$ is the *dual net* associated with A_+:

$$A_+^{\rm dual}(O) := A_+(O')' \tag{2.3}$$

where $A_+(O') := \bigvee_{\hat O \subset O'} A_+(\hat O) \equiv A_+(O_<) \vee A_+(O_>)$ is the algebra generated by all observables of A_+ localized in double-cones at space-like separation from O.

Remarks: 1. Both nets A_+ and $A_+^{\rm dual}$ are represented on the same Hilbert space $\mathcal{H}_0$, the vacuum Hilbert space of A. The observables of the trivial BCFT are bilocal expressions in the chiral observables, as described in the Introduction.

2. The dual net is local because, if O_1 and O_2 are space-like separated within M_+, then $L_2 \subset K_1$ (or $1 \leftrightarrow 2$), hence $A_+^{\rm dual}(O_1) \subset A(K_1)'$ and $A_+^{\rm dual}(O_2) \subset A(L_2)$ commute. It follows that $A_+^{\rm dual}$ is its own dual net (*Haag-duality*).

3. The inclusion

$$A_+(O) \subset A_+^{\rm dual}(O) \tag{2.4}$$

is the "two-interval subfactor" extensively discussed in [29]. Apart from $A(I)$ and $A(J)$, the algebra $A_+^{\rm dual}(O)$ contains all unitary "charge transporters" $u : \rho^I \to \rho^J$ where ρ^I, ρ^J are (equivalent) DHR endomorphisms of A localized in I and J, respectively,[3] and these elements generate $A_+^{\rm dual}(O)$. The algebraic isomorphism class of the two-interval subfactor (2.4) does not depend on the pair of intervals, and thus on O.

4. We observe that

$$\bigvee_{O:\, L_O \subset L} A_+(O) = \bigvee_{O:\, L_O \subset L} A_+^{\rm dual}(O) = A(L), \tag{2.5}$$

because the intervals I_O and J_O, as O varies as specified, cover all of L.

The trivial BCFT A_+ and its dual $A_+^{\rm dual}$ are special cases [4]) of boundary conformal quantum field theories in the sense of the following definition.

2.2 Definition: A *boundary CFT (BCFT)* associated with A is a local, isotonous net $O \mapsto B_+(O)$ over the double-cones within the half-space M_+, represented on a Hilbert space $\mathcal{H}_B$ such that

(i) there is a unitary representation $\mathcal{U}$ of the covering of the Möbius group $G = \widetilde{PSL}(2,\mathbb{R})$ with positive generator for the subgroup of translations, such

---

[3]For details on DHR theory in the chiral setting, see [15, 16]. The notation $t : \rho \to \sigma$ means the intertwining property $t\rho(a) = \sigma(a)t$ for all $a \in A$. We shall also write $t \in \mathrm{Hom}(\rho, \sigma)$.

[4]The latter is sometimes called "the Cardy case" in the literature [14].

that

$$\mathcal{U}(g)B_+(O)\mathcal{U}(g)^* = B_+(gO) \tag{2.6}$$

whenever the conformal transformation $g \in G$ takes the double-cone $O = I_O \times J_O$ within M_+ into another double-cone $gO := gI_O \times gJ_O$ within M_+ [5] (i.e., in particular for all translations and dilations), with a unique invariant vector $\Omega \in \mathcal{H}_B$ (the vacuum vector).

(ii) There is a representation π of A on $\mathcal{H}_B$ such that $B_+(O)$ contains $\pi(A_+(O))$, and

$$\mathcal{U}(g)\pi(A_+(O))\mathcal{U}(g)^* = \pi(A_+(gO)) \tag{2.7}$$

whenever O and gO are double-cones within M_+.

(iii) *"Joint irreducibility":* For each double-cone O, the von Neumann algebra $B_+(O) \vee \pi(A_+)''$ is irreducible on $\mathcal{H}_B$, i.e., equals $\mathcal{B}(\mathcal{H}_B)$. Here, $\pi(A_+)$ is the C* algebra generated by all double-cone algebras $\pi(A_+(O))$, and $\pi(A_+)''$ is its weak closure, i.e., the von Neumann algebra generated by all interval algebras $\pi(A(I))$.

Comments: 1. By Remark 4 following Def. 2.1, the covariance condition in (ii) is equivalent to

$$\mathcal{U}(g)\pi(A(I))\mathcal{U}(g)^* = \pi(A(gI)) \tag{2.8}$$

whenever I and gI are intervals in $\mathbb{R}$. As a consequence, π extends to a positive-energy representation of the chiral net A on the circle.

2. Joint irreducibility (iii) implies irreducibility of the net B_+ on $\mathcal{H}_B$ with Ω the unique $\mathcal{U}$-invariant vector, cf. [20]. On the other hand, Ω being the unique $\mathcal{U}$-invariant vector and cyclic for B_+ implies the irreducibility of B_+ by Prop. 3.3 below (choosing U the subgroup of time translations). The covariance and spectrum condition (i) implies that the vacuum vector is in fact cyclic and separating for every local algebra $B_+(O)$ (Reeh-Schlieder property).

3. Joint irreducibility also implies that the local inclusions $\pi(A_+(O)) \subset B_+(O)$ have trivial relative commutant.

4. Joint irreducibility is automatic if the representation $\mathcal{U}(g)$ belongs to $\pi(A_+)''$, e.g., if the stress-energy tensor of the BCFT coincides with that of the chiral theory A, see e.g., [39, 2].

5. In general, a BCFT net B_+ does *not* contain the dual net $\pi(A_+^{\mathrm{dual}})$, nor is it relatively local with respect to $\pi(A_+^{\mathrm{dual}})$ (see Prop. 2.7 for a characterization of this case; clearly, the former property would imply the latter).

[5] G being a covering group, this means more precisely the following: g is represented by a path $g_t \in PSL(2,\mathbb{R})$ connecting the identity with $p(g) \in PSL(2,\mathbb{R})$, such that g_tO lies within M_+ for all t.

2.3 Definition: The *dual net* $O \mapsto B_+^{\rm dual}(O)$ of a boundary CFT $O \mapsto B_+(O)$ is defined by

$$B_+^{\rm dual}(O) := B_+(O')' \equiv B_+(O_<)' \cap B_+(O_>)'. \tag{2.9}$$

Here, $B_+(O_<)$, $B_+(O_>)$ are the von Neumann algebras generated by all $B_+(\hat{O})$ as $\hat{O}$ belongs to the left or right causal complement of O, respectively. By locality of B_+, $B_+^{\rm dual}(O)$ contains $B_+(O)$.

We shall see below (Prop. 2.10, using *Modular Theory* [43]) that B_+ is in fact *wedge dual*, i.e., the algebra of a right wedge W_R is the commutant of the algebra of the corresponding left wedge $W_R = W_L'$, and vice versa. This means that (2.9) may be rewritten as

$$B_+^{\rm dual}(O) := B_+(O_>') \cap B_+(O_<'). \tag{2.10}$$

In particular, the dual net $B_+^{\rm dual}$ is again local, and consequently it is its own dual (i.e., it is Haag dual). The notation is consistent since $A_+^{\rm dual}$ is indeed the dual net of A_+.

2.3. The non-local chiral net associated with a BCFT.

We now turn to the description of a BCFT in terms of a *chiral* net, which is in general *non-local*.

2.4 Definition: A boundary CFT $O \mapsto B_+(O)$ generates a chiral net $I \mapsto B^{\rm gen}(I)$ (the associated *boundary net*) on $\mathcal{H}_B$, by

$$B^{\rm gen}(I) := \bigvee_{O \subset W_L} B_+(O) \equiv B_+(W_L) \tag{2.11}$$

where W_L is the left wedge spanned by I (cf. Fig. 4).

By the above Remark 4 following Def. 2.1, this definition associates the original net A with both A_+ and $A_+^{\rm dual}$.

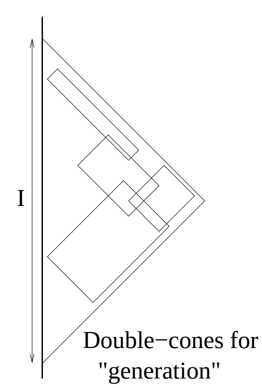

Fig. 4: The observables of the associated chiral boundary net localized in I are generated by BCFT observables localized in double-cones $O \subset W_L$.

2.5 Proposition: (i) The boundary net $B^{\rm gen}$ generated from B_+ is isotonous, and it is covariant:

$$\mathcal{U}(g)B^{\rm gen}(I)\mathcal{U}(g)^* = B^{\rm gen}(gI) \tag{2.12}$$

whenever $I \subset \mathbb{R}$, $gI \subset \mathbb{R}$ [6]. It acts irreducibly on $\mathcal{H}_B$. $B^{\rm gen}$ extends $\pi(A)$ and is relatively local with respect to $\pi(A)$:

$$\pi(A(I)) \subset B^{\rm gen}(I) \subset \pi(A(I'))'. \tag{2.13}$$

(ii) There is a consistent family of vacuum-preserving conditional expectations $\mathcal{E}^I : B^{\rm gen}(I) \to A(I)$.

(iii) The local subfactors $\pi(A(I)) \subset B^{\rm gen}(I)$ are irreducible and have finite index. The index is independent of I.

In general, the boundary net $B^{\rm gen}$ is a *non-local* chiral net. For if I_1 and I_2 are disjoint, then the double-cones contributing to the definition (2.11) of the corresponding algebras $B^{\rm gen}(I_1)$ and $B^{\rm gen}(I_2)$ are pairwise time-like separated, and observables of B_+ need not satisfy time-like commutativity. Non-local chiral nets have been studied before, e.g., in [1]. We shall review and extend their general structure theory in Sect. 3. In the remainder of the present section, we shall freely use these results.

Proof of Prop. 2.5: (i) Isotony, covariance, the extension property and relative locality are elementary. Irreducibility of the net $B^{\rm gen}$ follows from irreducibility of B_+.

(ii) The statement will be proven in the next section (Prop. 3.5(i)).

(iii) Irreducibility of the local subfactors $A(I) \subset B^{\rm gen}(I)$ follows from joint irreducibility. Namely, the von Neumann algebra $\pi(A(I)) \vee B^{\rm gen}(I)'$ contains $\pi(A(I))\vee\pi(A(I'))$ hence $\pi(A_+)$ by strong additivity, and $B_+(O)$ for some O with $I \subset K_O$, hence it equals $\mathcal{B}(\mathcal{H}_B)$. Thus $\pi(A(I))' \cap B^{\rm gen}(I) = \mathbb{C}\cdot\mathbf{1}$.

Irreducibility implies finiteness of the index because A is completely rational, by the same argument as in [28, Prop. 2.3] (i.e., each irreducible subsector ρ can arise in π with multiplicity bounded by $d(\rho)$). Its independence of the interval follows as in [33, Cor. 4.2]. Q.E.D.

Prop. 2.5 means that the extension $\pi(A) \subset B^{\rm gen}$ defines what was called a *quantum field theoretical net of subfactors* in [33]. We shall use here rather the terminology *chiral extension*. Conditional expectations having the abstract properties of an average ("non-commutative integration"), the existence of a consistent family of vacuum-preserving conditional expectations

[6] in the same sense as explained in the footnote to Def. 2.2(ii)

of $B(I)$ to $A(I)$ was viewed in [33] as a "generalized symmetry which is unbroken in the vacuum state".

It should be emphasized that a consistent family of (vacuum-preserving) conditional expectations cannot be expected in general for the double-cone algebras $A_+(O) \subset B_+(O)$, because the modular automorphism group of $B_+(O)$ acts non-geometrically and therefore does not preserve the subalgebra $A_+(O)$. In the case of $B_+ = A_+^{\rm dual}$, the failure can be seen directly: here the cyclic subspace of A_+ coincides with the full Hilbert space, and the corresponding projection is the unit operator. On the other hand, the unique conditional expectation of $A_+^{\rm dual}(O)$ to $A_+(O)$ [29] does not preserve the vacuum. Likewise, there cannot be a *global* conditional expectation, because, e.g., $\mathcal{E}^{\hat{O}}$ is trivial on $A_+^{\rm dual}(O)$ whenever $I_{\hat{O}}$ or $J_{\hat{O}}$ contains L_O because $A_+^{\rm dual}(O) \subset A(L_O) \subset A_+(\hat{O})$, while $\mathcal{E}^O$ is non-trivial on the same algebra.

In this sense, the (generalized) symmetry allows to determine the subalgebras $\pi(A_+(W)) \subset B_+(W)$ associated with wedges as fixpoint algebras, but the same does not hold for the subalgebras $\pi(A_+(O)) \subset B_+(O)$ associated with double-cones. (This is completely analogous to compact symmetry groups acting on field algebras associated with connected and with disconnected regions in four dimensions [12]). As a consequence, the techniques and results of [33] do not apply directly to algebraic boundary CFT, considered as the net of subfactors $I \mapsto A_+(O) \subset B_+(O)$. Instead, as a consequence of Prop. 2.5, these techniques *do* apply to the associated boundary extension $A(I) \subset B^{\rm gen}(I)$, and we shall elaborate in Sect. 4 and 5 how they *indirectly* provide the desired insight into the structure of the net of algebras on the half-space and its representations.

A central result of [33] is the following generation property (for further explanations, see Sect. 4 and App. A).

2.6 Corollary [33]: For each interval I, the "dual canonical" endomorphism of $A(I)$ associated with the local subfactor $A(I) \subset B^{\rm gen}(I)$ extends to a DHR endomorphism θ^I of A localized in I. The algebra $B^{\rm gen}(I)$ is generated by its subalgebra $\pi(A(I))$ and a "canonical" isometry $v^I \in B^{\rm gen}(I)$ which is an intertwiner for θ^I, i.e., one has $\pi(\theta^I(a))v^I = v^I\pi(a)$ for all $a \in A$.

This property can be used to obtain

2.7 Proposition: If B_+ is relatively local with respect to $\pi(A_+^{\rm dual})$, then $B^{\rm gen} = A$, and B_+ lies between A_+ and $A_+^{\rm dual}$.

Proof: Let $O = I \times J$, $J < I$, and K and L as described in the beginning of Sect. 2.1. Assume that $\pi(A_+^{\rm dual}(O))$ commutes with $B_+(\hat{O})$ whenever $\hat{O}$ belongs to the left causal complement of O, i.e., whenever $L_{\hat{O}} \subset K$. Then $\pi(A_+^{\rm dual}(O))$ commutes with $B^{\rm gen}(K)$. Every unitary charge trans-

porter $u : \rho^I \to \rho^J$ belongs to $A_+^{\rm dual}(O)$, thus $\pi(u)$ commutes with $B^{\rm gen}(K)$. By Cor. 2.6, $v^K \in B^{\rm gen}(K)$ satisfies $v^K\pi(u) = \pi(\theta^K(u))v^K$ while by locality $v^K\pi(u) = \pi(u)v^K$, hence $\pi(\theta^K(u))v^K = \pi(u)v^K$. As an equation in $B^{\rm gen}(L)$, this implies [33] $\theta(u) = u$. By [16], this implies that the sectors $[\rho]$ and $[\theta]$ have trivial monodromy, and as $[\rho]$ was arbitrary, $[\theta]$ has trivial monodromy with every DHR sector. But by [29], the braiding of the completely rational net A is non-degenerate, hence $[\theta]$ must be trivial. This in turn implies $B^{\rm gen} = A$ [33].

The last statement will follow from Prop. 2.9(ii), according to which $B^{\rm gen} = A$ implies $B_+^{\rm dual} = A_+^{\rm dual}$. Q.E.D.

By the definition of the boundary net B and locality of B_+, we obviously have $B_+(O) \subset B^{\rm gen}(L) \cap B^{\rm gen}(K)'$. This suggests the following definition of a local boundary CFT *induced* by a given (possibly non-local) chiral net:

2.8 Definition: If $I \mapsto B(I)$ is an irreducible chiral extension of $I \mapsto A(I)$ (possibly non-local, but relatively local with respect to A), then the *induced net* is defined by (cf. Fig. 5)

$$O \mapsto B_+^{\rm ind}(O) := B(L) \cap B(K)'. \tag{2.14}$$

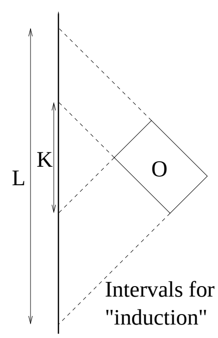

Fig. 5: The observables of the induced BCFT localized in O belong to $B(L)$ and commute with $B(K)$.

Let us discuss to which extent Def. 2.8 is the converse of Def. 2.4, i.e., to which extent a boundary CFT can be reconstructed from its boundary net:

2.9 Proposition: (i) The induced net (2.14) is a boundary CFT associated with A, defined on the Hilbert space of B. E.g., in the special case $B = A$, the induced net is the dual net $A_+^{\rm dual}$.

(ii) If B is a chiral extension of A, then the boundary net $(B_+^{\rm ind})^{\rm gen}$ generated by the induced net $B_+^{\rm ind}$ is again B. Conversely, if B_+ is a boundary CFT, then its boundary net $B^{\rm gen}$ induces the dual net $B_+^{\rm dual}$ associated with B_+, i.e., $(B^{\rm gen})_+^{\rm ind} = B_+^{\rm dual}$. In other words, we have: gen $\circ$ ind = id, ind $\circ$

gen = dual, implying dual $\circ$ ind = ind and dual $\circ$ dual = dual.

(iii) Every induced net $B_+^{\rm ind}$ is self-dual (Haag dual).

Proof: (i) $B_+^{\rm ind}$ contains $\pi(A_+)$ and is local by definition. The covariance properties (2.6) and (2.7) follow from covariance of the chiral net B. Joint irreducibility is automatic because $\pi(A(I)) \subset B(I)$ has finite index by virtue of irreducibility of $\pi(A(I)) \subset B(I)$ and complete rationality [28], which implies that $\mathcal{U}(g)$ belongs to the von Neumann algebra $\pi(A)''$ generated by all $\pi(A(I))$, cf. Comment 4 after Def. 2.2.

(ii) $(B_+^{\rm ind})^{\rm gen}(L)$ is generated by the algebras $B(L) \cap B(K)'$ as K varies within L, so its commutant is the intersection of the algebras $B(L)' \vee B(K)$ as K varies. For any fixed $K_0 \subset L$, by the split property for the net B (Prop. 3.6), $B(L)' \vee B(K)$ is naturally isomorphic to $B(L)' \otimes B(K_0)$. Now, as K varies within K_0, the intersection $\bigcap_{K\subset K_0} B(K)$ is trivial (this follows from "triviality at a point", Prop. 3.2(ii)), hence $\bigcap_{K\subset K_0} B(L)' \otimes B(K) = B(L)' \otimes \mathbb{C}\mathbf{1}$. It follows that $\bigcap_{K\subset K_0} B(L)' \vee B(K)$ equals $B(L)' \vee \mathbb{C}\mathbf{1} = B(L)'$, and $(B_+^{\rm ind})^{\rm gen}(L) = \bigvee_K B(L) \cap B(K)' = B(L)$.

Conversely, if B_+ is given, then by definition

$$B^{\rm gen}(K) = B_+(O_<) \tag{2.15}$$

since $O_<$ is the left wedge spanned by K. We shall show next (Prop. 2.10) that boundary CFT nets satisfy *wedge duality*. Hence, because the right wedge $O_>$ is the causal complement of the left wedge spanned by L, we have also

$$B_+(O_>) = B^{\rm gen}(L)'. \tag{2.16}$$

This implies $B_+^{\rm dual}(O) = B^{\rm gen}(L) \cap B^{\rm gen}(K)' = (B^{\rm gen})_+^{\rm ind}(O)$.

(iii) is obvious from (ii). Q.E.D.

As the examples (Def. 2.1) of the trivial BCFT and its dual show, there is no bijection between boundary CFT's and their boundary nets; but Prop. 2.9(ii) means that there is a bijection between *Haag-dual* boundary CFT's and their boundary nets. Yet, the non-Haag-dual boundary CFT's being subtheories of the Haag-dual ones, the previous results show that a classification of boundary CFT's essentially reduces to a classification of (non-local) *chiral* extensions.

The following facts (some of which anticipate results from Sect. 5) provide some non-trivial examples for the results in this section:

The chiral theory A of the stress-energy tensor with $c = \frac{1}{2}$ has one non-trivial chiral extension B, the CAR algebra of a chiral real Fermi field ψ on $\mathcal{H}_B = \mathcal{H}_0 \oplus \mathcal{H}_{\frac{1}{2}}$. The local algebras $A_+^{\rm dual}(O)$ of the dual net are generated by

$A_+(O)$, the operators $(\psi(f)\psi(g))\restriction_{\mathcal{H}_0}$ with supp $f \subset I$, supp $g \subset J$, and the field ϕ_0 (eq. (1.11)) smeared within O. (In Sect. 5 it will become clear that this characterization is equivalent with the one given in Remark 3 following Def. 2.1.) On the other hand, the local algebras $B_+^{\mathrm{ind}}(O)$ are generated by $\pi(A_+(O))$, $\psi(f)\psi(g)$ and the field ϕ_1 (eq. (1.12)). The subtheory on $\mathcal{H}_0$ with local algebras $B_+(O) = (CAR(I) \otimes CAR(J))^{\mathrm{even}}$ generated by $\pi(A_+)$ and $\psi(f)\psi(g)$ is not Haag dual. While the field ϕ_0 may well be "lifted" to $\mathcal{H}_{\frac{1}{2}}$, it is impossible to do so such that it locally commutes also with ϕ_1.

In Sect. 4, we shall show how to "compute" the intersection $B_+^{\mathrm{ind}}(O) = B(L) \cap B(K)'$ from the non-local chiral extension B of A in the general case (in terms of the DHR category of the local chiral net A), and to obtain algebraic invariants for the inclusions $\pi(A_+(O)) \subset B_+^{\mathrm{ind}}(O)$ from the chiral subfactors $\pi(A(I)) \subset B(I)$.

2.4. General results: duality and split property for wedges

We proceed with several general structure results about boundary CFT's.

In the sequel, for W a left wedge, Δ_W^{is}, J_W are the modular data associated with the von Neumann algebra $B_+(W)$ and the vacuum state [43, Chap. VI, Thm. 1.19], and $\Lambda_W(s)$ is the one-parameter subgroup of the (covered) Möbius group G preserving W, defined as follows: Let $\lambda(s) : u \mapsto \exp(-s)u$ be the scale transformations. Then $\Lambda_W(s) := g\lambda(s)g^{-1}$ where $g \in PSL(2,\mathbb{R})$ maps $(0,\infty)$ to the interval I of the real line which spans W.

We denote by r_W the inversion which maps I to its complement on the circle, i.e., $r_W = hrh^{-1}$ where r is the inversion $u \mapsto 1/u$, and $h \in G$ maps $(-1,1)$ to I. We denote by the same symbol the (densely defined) transformations of Minkowski space or the half-space M_+, induced by acting simultaneously on $t+x$ and $t-x$.

Finally, Γ is the group generated by r and G. Then we prove

2.10 Proposition: (i) Every boundary CFT B_+ satisfies *wedge duality*

$$B_+(W') = B_+(W)' \tag{2.17}$$

where W is a left wedge and W' its causal complement.

(ii) Every boundary CFT B_+ has the *Bisognano-Wichmann property*:

$$\Delta_W^{is} = \mathcal{U}(\Lambda_W(-2\pi s)) \tag{2.18}$$

for every left wedge W. There exists an (anti-)unitary representation $\tilde{\mathcal{U}}$ of the group Γ on $\mathcal{H}_B$ extending the representation $\mathcal{U}$ of G such that

$$J_W = \mathcal{U}(r_W). \tag{2.19}$$

In particular, $J_W \mathcal{U}(g) J_W = \mathcal{U}(r_W g r_W)$ $(g \in \Gamma)$ and

$$J_W B_+^{\mathrm{dual}}(O) J_W = B_+^{\mathrm{dual}}(r_W O) \tag{2.20}$$

whenever O and $r_W O$ are double-cones within M_+.

Proof: The first part (2.18) of (ii) is proven in Prop. 3.5(ii) where $B(I) \equiv B_+(W)$, $A(I) \equiv A_+(W)$, using 2.5(iii).

Turning to (i), we note that wedge duality holds for A_+, because it is equivalent to Haag duality on the real line for A, which is in turn equivalent to strong additivity.

Let W be a left wedge and W' its causal complement. Consider the inclusions

$$\pi(A_+(W')) \subset B_+(W') \subset B_+(W)'. \tag{2.21}$$

The subfactor

$$\pi(A_+(W')) = J_W \pi(A_+(W)) J_W \subset J_W B_+(W) J_W = B_+(W)' \tag{2.22}$$

is irreducible with finite index, because $\pi(A_+(W)) = \pi(A(I)) \subset B^{\mathrm{gen}}(I) = B_+(W)$ is irreducible with finite index. Clearly, $B_+(W)'$ is globally stable under $\mathrm{Ad}_{\mathcal{U}(\Lambda_W(s))}$, and the same is true for $A_+(W')$ by strong additivity of A. Due to the rigidity of intermediate subfactors in subfactors with finite index [32], the intermediate algebra $B_+(W')$ in (2.21) must also be globally stable under $\mathrm{Ad}_{\mathcal{U}(\Lambda_W(s))}$.

Thus, $B_+(W')$ is a von Neumann subalgebra of $B_+(W)'$ cyclic on the vacuum, which, thanks to (2.18), is in addition invariant under the modular automorphisms of $B_+(W)'$. By modular theory [43, Chap. IX, Thm. 4.2], this algebra must coincide with $B_+(W)'$. This proves wedge duality.

Turning to the second part of (ii), we infer from Prop. 3.2(iv) that $J_W \mathcal{U}(g) J_W = \mathcal{U}(r_W g r_W)$ $(g \in G)$, thus (2.19) defines a representation. Furthermore, Ad_{J_W} acts covariantly on interval algebras: $\mathrm{Ad}_{J_W} B^{\mathrm{gen}}(K) = B^{\mathrm{gen}}(r_W K)$, hence on left wedge algebras: $\mathrm{Ad}_{J_W} B_+(W_L) = B_+(r_W W_L)$, hence on right wedges by wedge duality: $\mathrm{Ad}_{J_W} B_+(W_R) = B_+(r_W W_R)$. As $B_+^{\mathrm{dual}}(O)$ is defined as an intersection of wedge algebras, it follows that Ad_{J_W} acts covariantly on the dual net as stated in (2.20). Q.E.D.

E.g., if W is spanned by the interval $(-1,1)$, then $r_W : u \mapsto 1/u$ induces the ray inversion $(t,x) \mapsto (\frac{t}{t^2-x^2}, -\frac{x}{t^2-x^2})$. This map maps only the region $x > |t|$ of M_+ into M_+. Thus, (2.20) makes sense only for double-cones $I \times J$ such that $I \subset \mathbb{R}_+$ and $J \subset \mathbb{R}_-$. On other double-cones, J_W acts non-geometrically.

Remark: If B_+ is not Haag-dual, one may consistently define $\bar{B}_+(O)$ by $J_W B_+(r_W O) J_W$ (choosing W such that $r_W O$ belongs to M_+). This defines another BCFT intermediate between $A_+(O)$ and $B_+^{\rm dual}(O)$.

2.11 Proposition: Every boundary CFT B_+ satisfies the split property for wedges. That is, if O is a double-cone within M_+ and $W_L = O_<$ and $W_R = O_>$ the associated pair of left and right wedges, then the inclusion $B_+^{\rm ind}(W_L) \subset B_+^{\rm ind}(W_R)'$ is split, or equivalently $B_+^{\rm ind}(W_L) \vee B_+^{\rm ind}(W_R)$ is naturally isomorphic to the tensor product $B_+^{\rm ind}(W_L) \otimes B_+^{\rm ind}(W_R)$. In particular, this implies the split property for double-cones O_1, O_2 whenever $O_1 \subset O_<$ and $O_2 \subset O_>$, i.e., $B_+^{\rm ind}(O_1) \vee B_+^{\rm ind}(O_2)$ is naturally isomorphic to the tensor product $B_+^{\rm ind}(O_1) \otimes B_+^{\rm ind}(O_2)$.

Proof: The inclusion $A \subset B^{\rm gen}$ has finite index (Prop. 2.5(iii)). Thus $B^{\rm gen}$ is split by Prop. 3.6, i.e., the inclusion $B^{\rm gen}(K) \subset B^{\rm gen}(L)$ is split. Now by definition, $B_+(W_L) = B^{\rm gen}(K)$, and by wedge duality (Prop. 2.10(i)), $B_+(W_R)' = B^{\rm gen}(L)$. This proves the claim. Q.E.D.

2.12 Proposition: Let B be a chiral extension of A, and $B_+^{\rm ind}$ the induced BCFT net. Then

(i) The index of $\pi(A_+(O)) \subset B_+^{\rm ind}(O)$ equals the μ-index μ_A of A (i.e., the index of the two-interval subfactor $\mu_A = [A(L') \cap A(K) : A(I) \vee A(J)] = [A_+^{\rm dual}(O) : A_+(O)]$ which coincides with the dimension of the DHR category of A [29]; in particular, it is independent of O). This index is thus the same for each chiral extension.

(ii) The induced net $B_+^{\rm ind}$ satisfies strong additivity.

Proof: (i) Let $\lambda = [B(I) : \pi(A(I))]$ denote the index of the chiral extension. λ is independent of I and finite (Prop. 2.5(iii)). We want to compute the index of $\pi(A(I) \vee A(J)) \subset B(K)' \cap B(L)$, which equals the index of the commutant $\pi(A(I))' \cap \pi(A(J))' \supset B(K) \vee B(L)'$.

Using the notation

$$N_1 \overset{\alpha}{\subset} N_2 \tag{2.23}$$

to indicate that a subfactor $N_1 \subset N_2$ has index $[N_2 : N_1] = \alpha$, we shall prove the indices displayed in the square of inclusions

$$\begin{array}{ccc} \pi(A(I))' \cap \pi(A(J))' & \overset{?}{\supset} & B(K) \vee B(L)' \\ {\scriptstyle \lambda^2\mu_A} \cup & & \cup {\scriptstyle \lambda} \\ \pi(A(K)) \vee \pi(A(L')) & \overset{\lambda}{\subset} & \pi(A(K)) \vee B(L)' \end{array} \tag{2.24}$$

from which the index of the subfactor in the top row follows to be μ_A by the multiplicativity of the index, as claimed in the statement.

The inclusion in the left column is the two-interval subfactor of the chiral net A in the representation π on $\mathcal{H}_B$, whose index has been computed in [29, Lemma 42] as follows: π is unitarily equivalent to a DHR endomorphism θ of A in its vacuum representation [33] (see also Sect. 4), where θ has dimension $d(\theta) = \lambda$; thus we may as well consider the subfactor $\theta(A(K) \vee A(L')) \subset \theta(A(I) \vee A(J))'$ on $\mathcal{H}_0$. Choosing θ to be localized in K, we get $\theta(A(K)) \vee A(L') \subset A(K) \vee A(L') \subset (A(I) \vee A(J))'$. The former inclusion has index $d(\theta)^2 = \lambda^2$, and the latter is the two-interval subfactor of index μ_A. Thus, the index in the left column equals $\lambda^2 \mu_A$.

The indices in the right column and bottom row $[B(K) : \pi(A(K))]$ and $[B(L)' : \pi(A(L'))]$, respectively, by the split property for B (Prop. 3.6). The former is λ by definition, while the latter equals λ because in $\pi(A(L')) \subset B(L)' \subset \pi(A(L))'$ the second inclusion has index λ by definition, while the total inclusion is the one-interval subfactor of A in the representation π which has dimension λ^2 [29, Lemma 42].

Thus the index in the top row equals μ_A.

The statement (ii) now follows exactly as in [32, Lemma 23]. Q.E.D.

2.5. Superselection structure of boundary CFT.

In the remainder of this section, we discuss DHR sectors (= superselection sectors in the sense of [12]) for Haag dual boundary CFTs. Müger has shown [36] that in Minkowski space-time, the split property for wedges implies the absence of nontrivial sectors. We obtain here a similar result on the half-space.

A DHR sector of a boundary CFT B_+ is defined as an equivalence class of positive-energy representations π subject to the selection criterium that π cannot be distinguished from the (defining) vacuum representation by measurements within the causal complement of any double-cone O within M_+. Assuming Haag duality for B_+, by standard arguments [12] one finds that superselection sectors can be represented by localized and transportable endomorphisms (DHR endomorphisms) ρ of the net B_+. This means that for any given double-cone O, ρ can be chosen within its unitary equivalence class to act like the identity map on the algebra of the causal complement $B_+(O') = B_+(O_<) \vee B_+(O_>)$, and the unitary charge transporters which intertwine equivalent such endomorphisms localized in different regions belong to B_+. [7]

[7] Unlike double-cones in Minkowski space-time, any given pair of double-cones O_1, O_2 within M_+ is not always contained in another double-cone within M_+. However, given O_1 and O_2, one can choose an auxiliary O_3 such that $O_1 \cup O_3$ and $O_2 \cup O_3$ are each contained

It is obvious that every DHR endomorphism of B_+ defines a localized and transportable endomorphism (in the obvious sense) of the boundary net $B^{\rm gen}$; but the converse is not true. E.g., the DHR endomorphisms ρ of the chiral net A (which is the boundary net generated by $A_+^{\rm dual}$) localized in, say, the interval K act non-trivially on the charge transporters "across" K [16]. However, such charge transporters do belong to $A_+^{\rm dual}(O_>)$, so ρ is not localized as an endomorphism of $A_+^{\rm dual}$. In fact, the following result shows that the dual net $A_+^{\rm dual}$, and in fact any Haag dual boundary CFT, does not possess any nontrivial DHR sectors at all.

Let $E = O_1 \cup O_2$ be the union of two causally disjoint double-cones within M_+ which do not touch (i.e., whose closures are disjoint); we may assume that O_1 belongs to the left causal complement of O_2, $O_1 < O_2$. Then the causal complement E' of E is the union of the left wedge $W_L = O_{1<}$, the right wedge $W_R = O_{2>}$ and a double-cone $O = O_{1>} \cap O_{2<}$. We consider the inclusion $B_+(E) \subset B_+(E')'$, where $B_+(E)$ and $B_+(E')$ are defined by additivity.

2.14 Proposition: If $B_+ \supset A_+$ is a boundary CFT net, then the index μ_{B_+} of the inclusion

$$B_+(E) \subset B_+(E')' \tag{2.25}$$

is independent of E and equals

$$\mu_{B_+} = [B_+^{\rm dual} : B_+]^3 = \left(\frac{\mu_A}{[B_+ : \pi(A_+)]}\right)^3 \tag{2.26}$$

where the indices of the extensions $[B_+^{\rm dual} : B_+] := [B_+^{\rm dual}(O) : B_+(O)]$ and $[B_+ : \pi(A_+)] := [B_+(O) : \pi(A_+(O))]$ are independent of O. In particular, $\mu_{A_+} = \mu_A^3$.

2.15 Corollary: (i) When B_+ is Haag dual, then $\mu_{B_+} = 1$, and B_+ satisfies Haag duality also for disconnected regions of the form $E = O_1 \cup O_2$ as above (i.e., (2.25) is an equality).

(ii) A Haag dual boundary CFT net B_+ has no nontrivial DHR sectors.

(iii) When B_+ is not Haag dual, then $B_+^{\rm dual}$ is a field net for B_+ in the sense of [12], i.e., for every sector of B_+ represented by a DHR endomorphism ρ, there is a nontrivial operator in $B_+^{\rm dual}$ which intertwines ρ with the identity.

Proof of the Proposition: The independence of the indices on the various regions (of a given topology) follows as in [29, 33].

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in some double-cone within M_+. The charge transporter from O_1 to O_2 may then be obtained as a composition of two charge transporters from O_1 to O_3 and from O_3 to O_2.

We denote by B the induced boundary net, and write $[B : A] =: \lambda$ and $[B_+ : \pi(A_+)] =: \lambda_+$. We shall show that

$$\begin{array}{ccc} B_+(E) & \overset{\mu_{B_+}}{\subset} & B_+(E')' \\ \lambda_+^2 \cup & & \cap\ \lambda^2\lambda_+ \\ \pi(A_+(E)) & \overset{\lambda^2\mu_A^3}{\subset} & \pi(A_+(E'))' \end{array} \tag{2.27}$$

which implies

$$\mu_{B_+} = \left(\frac{\mu_A}{\lambda_+}\right)^3 \tag{2.28}$$

by multiplicativity of the index:

Bottom row of (2.27): $A_+(E) = A(J_2) \vee A(J_1) \vee A(I_1) \vee A(I_2)$ is a four-interval algebra of the chiral net A, and so is $A_+(E')$ by strong additivity of A. Thus we have the four-interval subfactor in the representation π whose index is computed, as in Prop. 2.12, with the help of [29, Lemma 42] to be $d(\theta)^2\mu_A^3 = \lambda^2\mu_A^3$.

Left column of (2.27) $[B_+(E) : \pi(A_+(E))] = \mu_A^2$: We have

$$B_+(E) = B_+(O_1) \vee B_+(O_2) \supset \pi(A_+(E)) = \pi(A(O_1)) \vee \pi(A(O_2)) \tag{2.29}$$

so, by the split property for B_+ (Prop. 2.11), $[B_+(E) : \pi(A_+(E))] = [B_+(O_1) : \pi(A_+(O_1))] \cdot [B_+(O_2) : \pi(A_+(O_2))]$ where each factor equals λ_+.

Right column of (2.27) $[B_+(E') : \pi(A_+(E'))] = \mu_A$: The computation is analogous to the previous one, but here $E' = W_L \cup O \cup W_R$ has 3 connected components. The double-cone contributes a factor λ_+ as before, while the two wedges contribute a factor $[B_+(W) : \pi(A_+(W))] = [B(I) : \pi(A(I))] = \lambda$ each.

This proves the various indices in (2.27) and hence the formula (2.28). By Prop. 2.12, $\mu_A = [B_+^{\mathrm{dual}}(O) : \pi(A_+(O))] = [B_+^{\mathrm{dual}}(O) : B_+(O)][B_+(O) : \pi(A_+(O))]$ gives $[B_+^{\mathrm{dual}}(O) : B_+(O)] = \mu_A/\lambda_+$. This proves (2.26). Q.E.D.

Proof of the Corollary: The statement (i) is obvious from the proposition. The proof for the absence of non-trivial sectors in the Haag dual case is exactly as (iii) $\Rightarrow$ (ii) in [29, Cor. 32][8], using Haag duality of B_+ for disconnected regions of the form E: If $u : \rho_1 \to \rho_2$ is a unitary intertwiner from ρ_1 localized in O_1 to ρ_2 localized in O_2, then u belongs to $B_+(E')' = B_+(E)$.

[8] There is some unfortunate misnumbering of the implications proven in [29, Cor. 32]. (i) $\Rightarrow$ (ii) should read (iii) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) should read (i) $\Rightarrow$ (iii), while (iii) $\Rightarrow$ (i) is trivial.

Thanks to the split property for B_+ (Prop. 2.11), there is a conditional expectation $\mathcal{E} : B_+(E) = B_+(O_1) \vee B_+(O_2) \to B_+(O_1)$ such that $\mathcal{E}(u) \neq 0$. This is a nontrivial local intertwiner from $\rho_1 \restriction B_+(O_1)$ to $\mathrm{id} \restriction B_+(O_1)$, hence a global intertwiner from ρ_1 to id thanks to strong additivity for B_+ (Prop. 2.12). Thus every sector contains the identity sector, which implies the claim.

Similarly, when B_+ is not Haag dual, and ρ_i are a pair of equivalent DHR endomorphisms of B_+ as before, then the charge transporter $u : \rho_1 \to \rho_2$ belongs to $B_+^{\mathrm{dual}}(E)$, and $\mathcal{E}(u) : \rho_1 \to \mathrm{id}$ belongs to $B_+^{\mathrm{dual}}(O_1)$, as asserted. Q.E.D.

Remark: To prevent misconceptions of the statement (ii) of Cor. 2.15, it should be pointed out that a Haag-dual BCFT can well have non-trivial positive energy-representations; e.g., every positive-energy representation of A defines a positive-energy representation of A_+^{dual}. But these representations are not localized in double-cones as required for DHR representations.

3 Non-local chiral CFT

This section reviews and generalizes known structural theorems about non-local chiral CFT, and also contains several new results. While the section is logically independent of BCFT, its results bear important implications for BCFT. They are freely used in other sections.

3.1 General structure: Covariance and modular symmetry.

In [1], the covariant transformation law for a non-local chiral CFT

$$\mathcal{U}(g)B(I)\mathcal{U}(g)^* = B(gI) \qquad (g \in G) \tag{3.1}$$

was assumed to hold *globally*, i.e. for every interval of the circle and without restriction on $g \in G$. It was shown that this implies the rotation of the circle by 4π to be represented by $\mathcal{U}(4\pi) = \mathbf{1}$, hence the conformal Hamiltonian L_0 has half-integer spectrum and the net B is (at least weakly) graded local.

In our setting, this restriction is too narrow. Depending on the spectrum of L_0 on $\mathcal{H}_B$, (3.1) holds for the induced net only locally as indicated in (2.12), admitting "more non-local" induced boundary CFT's than graded local ones. In order to generalize the analysis in [1], we first note

3.1 Lemma: Let $I \to B(I)$ be a net of von Neumann algebras defined on the intervals $I \subset \mathbb{R}$, and $\mathcal{U}$ a representation of $G = \widetilde{PSL}(2,\mathbb{R})$ on the same Hilbert space such that (3.1) holds whenever I and gI belong to $\mathbb{R}$.

Then, identifying $\mathbb{R}$ with $S^1 \setminus \{-1\}$ by means of a Cayley transformation, B extends to a net defined on the intervals of the (universal) covering $\mathcal{S}$ of S^1 for which (3.1) holds globally.

Sketch of the Proof: Use (3.1) as a definition of the algebra $B(gI)$ on the right-hand side whenever the conditions on I and g are *not* met, i.e., whenever gI belongs to $\mathcal{S}$ but not to the natural embedding of $\mathbb{R}$ into $\mathcal{S}$. Validity of (3.1) in the restricted sense ensures that this definition is consistent. Q.E.D.

Depending on the theory, the resulting net may enjoy a periodicity of the form $B(I + N \cdot 2\pi) = B(I)$ for some $N \in \mathbb{N}$, in which case it may as well be considered as a net on the N-fold covering of S^1. E.g., $N = 1$ if B is local, and $N = 2$ if it is $\mathbb{Z}_2$-graded local.

To the theory on the covering, the analysis of [20, 1] may be applied, giving the same conclusions except those which assume that the rotation by 2π takes an interval into itself and hence $\mathrm{Ad}_{\mathcal{U}(2\pi)}$ is an automorphism of $B(I)$; e.g., the above-mentioned triviality of $\mathcal{U}(4\pi)$. Thus, we have

3.2 Proposition [20, 1]: Let $I \mapsto B(I)$ be a chiral net defined on a covering $\mathcal{S}$ of the circle, satisfying the standard assumptions: $B(I)$ are von Neumann algebras on a Hilbert space $\mathcal{H}$, $I_1 \subset I_2$ implies $B(I_1) \subset B(I_2)$, there is a unitary representation $\mathcal{U}$ of $G = \widetilde{PSL}(2,\mathbb{R})$ on $\mathcal{H}$ such that (3.1) holds globally on $\mathcal{S}$, the rotation subgroup has a positive generator, and there is a $\mathcal{U}$-invariant vector $\Omega \in \mathcal{H}$ (the vacuum) cyclic for $\bigvee_I B(I)$ and separating for $\bigcap_I B(I)$. Then one has

(i) *Reeh-Schlieder property*: Ω is cyclic and separating for each $B(I)$.

(ii) *Irreducibility and Triviality at a point*: $\bigvee_I B(I) = \mathcal{B}(\mathcal{H})$ and $\bigcap_I B(I) = \bigcap_{I \ni x} B(I) = \mathbb{C}\mathbf{1}$.

(iii) *Additivity and Continuity*: if I and I_k are open intervals such that $I \subset \bigcup_k I_k$, then $B(I) \subset \bigvee B(I_k)$, and if $\bar{I}$ denotes the closure of I and $\bar{I} \supset \bigcap_k I_k$, then $B(\bar{I}) \supset \bigvee B(I_k)$.

(iv) *Modular Covariance*: For any interval $I \subset \mathcal{S}$, the modular automorphisms $\mathrm{Ad}_{\Delta_I^{it}}$ of the von Neumann algebra $B(I)$ with respect to the vacuum vector Ω [43, Chap. IV, Thm. 1.19] act geometrically by

$$\Delta_I^{it} B(J) \Delta_I^{-it} = B(\Lambda_I(-2\pi t)J) \qquad (J \subset \mathcal{S}, t \in \mathbb{R}) \tag{3.2}$$

where Λ_I is the I-preserving one-parameter subgroup of G which is conjugate to the scale transformations of $\mathbb{R}$. Moreover, if r is the reflection of $\mathcal{S}$ induced by $x \mapsto -x$, and Γ the group generated by G and r, then $\mathcal{U}$ extends to an (anti-)unitary representation of Γ by setting $\mathcal{U}(r_I) = J_I$, where J_I is

the modular conjugation of $(B(I),\Omega)$ and r_I is the unique reflection in Γ conjugate to r which has the boundary points of I as fixpoints.

(v) The unitaries

$$z(t) := \mathcal{U}(\Lambda_I(2\pi t))\Delta_I^{it} \tag{3.3}$$

do not depend on I and form a one-parameter group in the center of the gauge group[9].

(vi) *Bisognano-Wichmann property*: Provided B is local or $\mathbb{Z}_2$-graded local (fermionic)[10], then the central cocycle $z(t)$ in (v) is trivial:

$$z(t) = 1, \qquad \text{i.e.,} \qquad \Delta_I^{it} = \mathcal{U}(\Lambda_I(-2\pi t)). \tag{3.4}$$

(vii) If $z(t) = \mathbf{1}$, then one has the following equivalences: $\mathbb{C}\Omega$ are the only $\mathcal{U}$-invariant vectors $\Leftrightarrow$ $B(I)$ are factors $\Leftrightarrow$ B is irreducible, i.e., $\bigvee_I B(I) = \mathcal{B}(\mathcal{H})$ $\Leftrightarrow$ $\bigcap_I B(I) = \mathbb{C}\mathbf{1}$. In this case, if $B(I) \neq \mathbb{C}\mathbf{1}$, the factors are of type III_1.

Proof: As in [20, 1]. (ii) is proved by various instances of the subsequent Prop. 3.3: Choosing $M = \bigvee_I B(I)$ and U the subgroup of translations, gives irreducibility. Choosing $M = \bigvee_I B(I)'$ and U the translations, gives $\bigcap_I B(I) = \mathbb{C}\cdot\mathbf{1}$. Choosing $M = M_x \equiv \bigvee_{I\ni x} B(I)$ and U the subgroup of special conformal transformations preserving the point x, gives triviality at the point. (Note that every vector which is invariant under the time translations or under the special translations is automatically also invariant under the full conformal group, and hence is a multiple of Ω. Note also that isotony ensures invariance of M_x under the special conformal transformations although their action does not preserve $\mathbb{R}$.) Q.E.D.

We have used

3.3 Proposition: Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$, v a cyclic vector and U a one-parameter unitary group implementing automorphisms of M. If U has a positive generator, and v is the unique U-invariant vector, then $M = \mathcal{B}(\mathcal{H})$.

Proof: Let E denote the projection on $\mathbb{C}\cdot v$. Because E is one-dimensional and v is cyclic, the algebra $E \vee M$ generated by E and M contains every one-dimensional projection, hence coincides with $\mathcal{B}(\mathcal{H})$. On the other hand, by positivity of the generator [8], the spectrum condition implies that U belongs to M, and consequently its spectral projection E also belongs to M. Hence $E \vee M$ equals M. Q.E.D.

[9] The gauge group consists of all unitaries V on $\mathcal{H}_B$ such that $V\Omega = \Omega$ and $VB(I)V^* = B(I)$ for all I.

[10] This assumption will be substantially relaxed in the next subsection (Prop. 3.5).

For later use, we record a simple fact.

3.4 Proposition: If $I \mapsto B(I)$ is a non-local net, then a net $I \mapsto C(I) \subset B(I)$ relatively local with respect to B is local. There is a unique maximal such net. The maximal net is covariant under the covariance group of B, and its local algebras $C(I)$ are globally stable under the gauge group of B.

Proof: Locality of C is obvious. Existence and uniqueness of the maximal net hold because any two nets C_1 and C_2 within B and relatively local with respect to B generate $C_1 \vee C_2$ with the same properties. The stability and covariance statements follow from uniqueness.

3.2 Non-local extensions.

In this subsection we assume that the chiral net $I \mapsto B(I)$ contains a covariant net of subfactors $I \mapsto \pi(A(I)) \subset B(I)$ which is relatively local with respect to B (in particular, A is local). Then we have

3.5 Proposition: (i) There is a family of vacuum-preserving conditional expectations $\mathcal{E}^I : B(I) \to A(I)$.

(ii) If the local subfactors $\pi(A(I)) \subset B(I)$ are irreducible with finite index, then the central cocycle (3.3) is trivial, $z(t) = 1$, i.e., the Bisognano-Wichmann property (3.4) holds.

(iii) If $z(t) = 1$, then the family of local conditional expectations is consistent, i.e., $\mathcal{E}^I$ restricts to $\mathcal{E}^{\hat{I}}$ on $B(\hat{I})$ whenever $\hat{I} \subset I$, so that there is a global vacuum-preserving conditional expectation $\mathcal{E} : B \to A$ which maps $B(I)$ onto $A(I)$. $\mathcal{E}$ is implemented by the projection onto the cyclic subspace $\overline{\pi(A)\Omega}$.

Proof: (i) Consider the maximal net $I \mapsto C(I)$ given by Prop. 3.4. By Props. 3.2(v) and 3.4, $C(I)$ is globally stable under the modular automorphism group [43, Chap. VI, Thm. 1.19] associated with $B(I)$ and the vacuum. By Takesaki's Theorem [43, Chap. IX, Thm. 4.2], there exists a vacuum-preserving conditional expectation of $B(I)$ onto $C(I)$.

On the other hand, because C is local, we may apply [1] or Prop. 3.2(vi) to the net C to conclude that $z(t)$ is trivial on the cyclic subspace of C. Because $\mathcal{U}$ restricts to the covariance representation of C which in turn restricts to that of A, $A(I)$ is globally stable under the modular automorphism group (3.4) associated with $C(I)$ and the vacuum, so there is a vacuum-preserving conditional expectation of $C(I)$ onto $A(I)$. Composition of the two expectations gives an expectation $\mathcal{E}^I$ of $B(I)$ onto $A(I)$.

(ii) By Prop. 3.2, the central cocycle $z(s)$ given by (3.3) is a vacuum-preserving unitary one-parameter group on $\mathcal{H}_B$ whose adjoint action globally

preserves $B(I)$. We have to show that $z(s)$ is trivial.

Because there is a vacuum-preserving conditional expectation of $B(I)$ onto $A(I)$, the modular automorphisms of $B(I)$ restrict [43] to the modular automorphisms of $A(I)$ and the vacuum. Because A is local, $z(s)$ is trivial on the cyclic subspace of $A(I)$ (the vacuum subrepresentation of A in π_B). Hence $\mathrm{Ad}_{z(s)}$ is a one-parameter group of automorphisms of $B(I)$ acting trivially on $\pi(A(I))$. Thus, its fix-point subalgebra $B(I)^z$ is intermediate between $\pi(A(I))$ and $B(I)$, and the index $[B(I) : B(I)^z]$ is finite because $[B(I) : \pi(A(I))]$ is finite. But the fix-point index is the order of the quotient group of $\mathbb{R}$ by the subgroup which acts trivially on $B(I)$. This number can be either 1 or ∞. The latter being excluded, the fix-point subalgebra must be all of $B(I)$. Hence, the automorphic action of $\mathrm{Ad}_{z(s)}$ is trivial, i.e., $z(t)$ commutes with $B(I)$. Since the vacuum is cyclic for $B(I)$ and $z(s)$ preserves the vacuum, $z(s)$ itself must be trivial. This proves (ii).

(iii) Because $z(t) = 1$, the modular automorphism group of $B^{\mathrm{gen}}(I)$ coincides with the subgroup of Möbius transformations preserving I, hence it globally preserves $A(I)$. Again by Takesaki's Theorem [43, Chap. IX, Thm. 4.2], it follows that $\mathcal{E}^I$ is implemented by the projection on the subspace $\overline{\pi(A(I))\Omega}$. By the Reeh-Schlieder Theorem, this projection does not depend on I. This implies consistency. Q.E.D.

In the remainder of this subsection, we shall explain the characterization of non-local chiral extensions $I \mapsto B(I) \supset A(I)$ in terms of Q-systems within the DHR category of the chiral net $I \mapsto A(I)$.

In [33], a structural analysis of local and non-local extensions of quantum field theories in the algebraic framework has been developed. The main tool was the notion of a Q-system [31], characterizing a subfactor $N \subset M$ of finite index. For a brief review on Q-systems, see App. A.

A Q-system consists in a set of algebraic relations which, in the case of quantum field theory, amount to the statement that the (non-local) fields of the extension form a closed algebra under multiplication and conjugation, and satisfy local commutation relations with the chiral fields. This interpretation of a Q-system is made more transparent if the relations are reformulated in terms of charged intertwiners (cf. App. A).

The central result in [33] is that a Q-system (θ, w, x) *within the DHR category* of a local net A determines a relatively local net B which extends A: θ is required to be a DHR endomorphism of the net $O \mapsto A(O)$ localized in some region O_0, and consequently the isometries w and x belong to $A(O_0)$. The Q-system therefore determines a positive-energy representation $\pi \simeq \theta$ of the net A on a Hilbert space $\mathcal{H}_B$, and the local subfactor $\pi(A(O_0)) \subset B(O_0)$

on $\mathcal{H}_B$. The latter can then be "transported" to a covariant net of subfactors $O \mapsto [\pi(A(O)) \subset B(O)]$ equipped with a consistent family of conditional expectations preserving the vacuum. (Imposing the additional eigenvalue condition $\varepsilon(\theta,\theta)x = x$ would ensure $O \mapsto B(O)$ to be a *local* net.)

In the case of (non-local) chiral extensions $A(I) \subset B(I)$ at hand, A being completely rational implies that only finitely many (equivalence classes of) endomorphisms θ can appear in an irreducible Q-system: the argument is as in [28, Prop. 2.3], using the fact that the multiplicity n_s of each irreducible subsector $[\rho_s]$ of θ is bounded by the square of its dimension [24, p. 39]. In particular, the index of the local subfactors $A(I) \subset B(I)$ is finite (and so the stronger bound $n_s \leq d(\rho_s)$ [33, Cor. 4.6] applies). Moreover, it was shown in [23, Thm. 2.4] that each θ can arise only in finitely many inequivalent Q-systems. This means that the classification problem of Q-systems in the DHR category of a (completely) rational CFT is a finite problem with finitely many solutions, and thus, fixing A, there exist only finitely many non-local chiral extensions B.

Examples for Q-systems within the DHR category of a local net were given for local and non-local chiral extensions of chiral nets, and for local two-dimensional extensions of subnets $A_L \otimes A_R$ consisting of two (left and right) chiral nets [33]. The main result in [40] is that there is a systematic way (the *α-induction construction*, using results of [6]) to associate a local Q-system, and hence a local two-dimensional extension B_2^α of $A_2 = A \otimes A$, with any given chiral extension B of A.

3.3 The split property.

Let us now turn to the split property, which is related to phase space properties (existence of $\mathrm{Tr}\exp -\beta L_0$) in QFT [9, 1]. A commuting pair of von Neumann algebras (M_1, M_2) is split if there is a natural isomorphism from $M_1 \vee M_2$ to $M_1 \otimes M_2$, where $M_1 \vee M_2$ denotes the von Neumann algebra generated by M_1 and M_2. A chiral net B is split if the pair $(B(K), B(L)')$ is split whenever the open interval L contains the closure of the interval K. We want to prove "upward hereditarity" of the split property.

3.6 Proposition: Let B be a Möbius covariant net on S^1 and A a finite index subnet such that B is relatively local with respect to A. If A is split, then also B is split.

Note that B is possibly non-local, but relative locality implies that A is local. In the case of a *local* net B, the result was proven in [32].

To prepare the proof of Prop. 3.6, we need Prop. 3.7 and Lemma 3.8:

3.7 Proposition: Let M_1, M_2 be commuting factors and $N_k \subset M_k$ finite index subfactors, $k = 1, 2$. If the pair (N_1, N_2) is split, then also the pair (M_1, M_2) is split.

Let $\gamma_k : M_k \to N_k$ be canonical endomorphisms and (γ_k, T_k, S_k) the associated Q-systems. Then $M_k = N_k T_k$, and every $m^{(k)} \in M_k$ can be written as $m^{(k)} = n^{(k)} T_k$ where $n^{(k)} \in N_k$ is given by $n^{(k)} = \lambda_k \cdot \mathcal{E}_k(m^{(k)} T_k^*)$, with $\mathcal{E}_k$ the associated expectation from M_k to N_k and λ_k is the index $[M_k : N_k]$. Thus $||n^{(k)}|| \leq \lambda_k ||m^{(k)}||$.

3.8 Lemma: With the above notations we have $NT_1T_2 = M$ where $M = M_1 \vee M_2$, $N = N_1 \vee N_2$.

Moreover there is a constant $C > 0$ such that if $m \in M$ then $m = nT_1T_2$, with $n \in N$ and $||n|| \leq C||m||$.

Proof of Lemma 3.8: We first show that the second part of the statement with $m \in M_1 \cdot M_2$. Here $M_1 \cdot M_2$ is the product of M_1 and M_2 which is naturally isomorphic to the algebraic tensor product $M_1 \odot M_2$ by the Murray-von Neumann factorization lemma.

Let $m = \sum m_i^{(1)} m_i^{(2)}$ with $m_i^{(1)} \in M_1, m_i^{(2)} \in M_2$ and write $m_i^{(1)} = n_i^{(1)} T_1$, $m_i^{(2)} = n_i^{(2)} T_2$, with $n_i^{(k)} \in N_k$.

The subfactor $N_1 \otimes N_2$ of $M_1 \otimes M_2$ has finite index and the associated Q-system is the tensor product Q-system, hence there is a constant $C > 0$ such that $|| \sum n_i^{(1)} \otimes n_i^{(2)} || \leq C || \sum m_i^{(1)} \otimes m_i^{(2)} ||$ where the norms here are the spatial tensor product norms. Hence we have

$$|| \sum n_i^{(1)} n_i^{(2)} || = || \sum n_i^{(1)} \otimes n_i^{(2)} || \leq C || \sum m_i^{(1)} \otimes m_i^{(2)} || \leq C || \sum m_i^{(1)} m_i^{(2)} ||, \tag{3.5}$$

where the first equality holds because of the split property for (N_1, N_2) and the last inequality due to the minimality of the spatial tensor product norm.

Now we prove the general statement. Let $m \in M$ with $||m|| \leq 1$ and choose by Kaplanski density theorem a net of elements $m_j \in M_1 \cdot M_2$, With $||m_j|| \leq 1$ and $m_j \to m$ weakly.

We can write $m_j = n_j T_1 T_2$ where $n_j \in N$ and $||n_j|| \leq C$. With n a weak limit point of n_j, we then have $m = nT_1T_2$ and $||n|| \leq C$. Q.E.D.

Proof of Prop. 3.7: Let $\Phi : M_1 \otimes M_2 \to M$ be the linear map

$$m \mapsto \Phi(m) \equiv \Phi_0(n) T_1 T_2 \tag{3.6}$$

where $n \in N_1 \otimes N_2$ is the unique element such that $m = n \cdot T_1 \otimes T_2$ and $\Phi_0 = \Phi \restriction_{N_1 \otimes N_2}$ is the natural isomorphism of $N_1 \otimes N_2$ with $N_1 \vee N_2$. By the Lemma, Φ is surjective.

We show that Φ is multiplicative and respects the $*$ operation. First note that if $n \in N$ then

$$T_1 T_2 n = \theta(n) T_1 T_2 \tag{3.7}$$

where θ is the endomorphism of N which is transformed to $\gamma_1 \restriction_{N_1} \otimes \gamma_2 \restriction_{N_2}$ under Φ_0 (check this with $n \in N_1 \cdot N_2$, then it holds for all $n \in N$ by continuity).

Let $m' \in M_1 \otimes M_2$, $m' = n' \cdot T_1 \otimes T_2$ with $n' \in N_1 \otimes N_2$. Then

$$\begin{aligned} mm' = n \cdot T_1 \otimes T_2 \cdot n' \cdot T_1 \otimes T_2 = n\gamma_1 \otimes \gamma_2(n') \cdot T_1^2 \otimes T_2^2 = \\ = n\gamma_1 \otimes \gamma_2(n') \cdot \lambda_1 \mathcal{E}_1(T_1^2 T_1^*) T_1 \otimes \lambda_2 \mathcal{E}_2(T_2^2 T_2^*) T_2 \;, \end{aligned} \tag{3.8}$$

Thus, suppressing the symbol Φ_0 for simplicity,

$$\Phi(mm') = n\theta(n') \cdot \lambda_1 \mathcal{E}_1(T_1^2 T_1^*) \lambda_2 \mathcal{E}_2(T_2^2 T_2^*) \cdot T_1 T_2 = n\theta(n') T_1^2 T_2^2 \;. \tag{3.9}$$

On the other hand

$$\Phi(m)\Phi(m') = n T_1 T_2 n' T_1 T_2 = n\theta(n') T_1^2 T_2^2 \tag{3.10}$$

as desired. As for the $*$ operation, the argument is completely analogous, using the formula $T_i{}^* = \lambda_i \mathcal{E}_i(T_i{}^{*2}) T_i$.

Thus, Φ is a *-homomorphism of von Neumann algebras, hence σ-weakly continuous. Since $M_1 \otimes M_2$ is a factor, Φ is injective and (M_1, M_2) is a split pair. Q.E.D.

Proof of Prop. 3.6. Let $I \subset\subset \tilde{I}$ be intervals and apply Prop. 3.7 with $N_1 = A(I)$, $N_2 = A(\tilde{I}')$, $M_1 = B(I)$, $M_2 = B(\tilde{I})'$. We just note that $[M_2 : N_2] < \infty$ because the inclusion $A(\tilde{I}') \subset B(\tilde{I})'$ is anti-isomorphic to

$$J_{\tilde{I}} A(\tilde{I}') J_{\tilde{I}} = A(\tilde{I}) \subset J_{\tilde{I}} B(\tilde{I})' J_{\tilde{I}} = B(\tilde{I}) \tag{3.11}$$

where $J_{\tilde{I}}$ is the modular conjugation of $B(\tilde{I})$ with respect to the vacuum and we are using the geometric action of $J_{\tilde{I}}$. Q.E.D.

4 Charged intertwiners in boundary CFT

The main result of this section is a generalization of [27, Thm. 3.1 and Remark 3.2] (where B was assumed to be local):

4.1 Theorem: For a given completely rational chiral net $I \mapsto A(I)$, and a given irreducible (possibly non-local) chiral extension $I \mapsto B(I)$, let

$O \mapsto B_+^{\mathrm{ind}}(O)$, $O = I \times J \subset M_+$, be the induced Haag dual boundary CFT net (Def. 2.8), and let $O \mapsto B_2^\alpha(O)$, $O = I \times J \subset M$, be the two-dimensional local net on Minkowski space extending $A \otimes A$, obtained from B by the α-induction construction. Then the local subfactors

$$A(I) \vee A(J) \subset B_+^{\mathrm{ind}}(O) \qquad \text{and} \qquad A(I) \otimes A(J) \subset B_2^\alpha(O) \tag{4.1}$$

are isomorphic.

Because $B_+(O)$ is intermediate between $A_+(O)$ and the dual net $B_+^{\mathrm{dual}}(O)$, we conclude

4.2 Corollary: For a given boundary CFT $O \mapsto B_+(O)$, let $O \mapsto B_2^\alpha(O)$ be the local two-dimensional extension of $A \otimes A$ obtained by applying the α-induction construction to the boundary net B of B_+. Under the isomorphism established in Thm. 4.1, the net $O \mapsto B_+(O)$ corresponds to an intermediate net

$$A(I) \otimes A(J) \subset B_2(O) \subset B_2^\alpha(O), \tag{4.2}$$

with $B_2(O) = B_2^\alpha(O)$ if and only if B_+ satisfies Haag duality.

In the course of the proof of the theorem, we shall "compute" the relative commutant $B_+^{\mathrm{ind}}(O) = B(L) \cap B(K)'$ by determining local operators $\psi_i \in B_+^{\mathrm{ind}}(O)$ (*charged intertwiners*, see below) which along with $A_+(O)$ generate $B_+^{\mathrm{ind}}(O)$. These charged intertwiners, as O varies, are the von Neumann analog of the non-chiral local Wightman fields of the boundary CFT, generalizing (1.11), (1.12). In Sect. 5, we shall further analyse their bi-localized charge structure.

In a general subfactor setup (see App. A for more details), the charged intertwiners for a subfactor $N \subset M$ are nontrivial elements ψ_i of M satisfying

$$\psi_i n = \varrho_i(n)\psi_i \qquad (n \in N) \tag{4.3}$$

(where ϱ_i are irreducible endomorphisms of N), such that every element of M has a unique expansion

$$m = \sum_i n_i \psi_i, \qquad (n_i \in N). \tag{4.4}$$

The *algebra of the charged intertwiners* in M

$$\psi_i \psi_j = \sum_k \Gamma_{ij}^k \, \psi_k, \qquad \psi_i^* = \sum_j \Gamma_{ji}^{0*} \, \psi_j. \tag{4.5}$$

with intertwiners $\Gamma_{ij}^k : \varrho_k \to \varrho_i \varrho_j$ in N, together with some normalization conditions, is an invariant of the subfactor $N \subset M$, determined by the Q-system.

Generalizing an argument used in [29], we show in Lemma A.2 that the Q-systems (γ, v, w) in M and (θ, w, x) in N can be recovered from the system of charged intertwiners $\psi_i \in M$; in particular, two such systems $\psi_i \in M$, $\tilde{\psi}_i \in \tilde{M}$ satisfying the same algebra with the same $\varrho_i \in \mathrm{End}(N)$ and $\Gamma_{ij}^k \in N$ induce an isomorphism of subfactors $N \subset M$, $N \subset \tilde{M}$ by $\psi_i \leftrightarrow \tilde{\psi}_i$.

Applying this argument to $N = A(I) \vee A(J) \simeq A(I) \otimes A(J)$ and $M = B_+^{\mathrm{ind}}(O)$, $\tilde{M} = B_2^\alpha(O)$, the statement of the theorem thus follows from the equivalence of the algebras of charged intertwiners which generate the respective inclusions.

First part of the proof of the Theorem: We proceed in close analogy with the proof of [29, Prop. 45]. $A_2(O) = A(I) \otimes A(J)$ and $A_+(O) = A(I) \vee A(J)$ are naturally isomorphic by the split property of the chiral net A. Under this isomorphism, the Q-system (Θ_2, W_2, X_2) for $A_2(O) \subset B_2^\alpha(O)$ (given in [40], see below) turns into a Q-system (Θ, W, X) in $A_+(O)$ with

$$X = d(\Theta)^{-\frac{1}{2}} \sum_{ijk} W^i \Theta(W^j)\, \Gamma_{ij}^k\, W^{k*}. \tag{4.6}$$

By the preceding discussion and Lemma A.2, it is sufficient to show that Θ coincides with the dual canonical endomorphism Θ_+ for $A_+(O) \subset B_+^{\mathrm{ind}}(O)$, and to find charged intertwiners $\psi_i \in B_+^{\mathrm{ind}}(O)$ satisfying the algebra (A.4–6+9) with Γ_{ij}^k given by (4.10).

Without loss of generality, we choose $I = (y, z) \subset \mathbb{R}_+$ and $J = -I \subset \mathbb{R}_-$ (this situation can always be attained by a conformal transformation), thus $K = (-y, y)$, $L = (-z, z)$ are symmetric, and put $O = I \times J$. Then $A(I) = j(A(J))$ where $j = \mathrm{Ad}_J$ is the modular conjugation [43, Chap. IV, Thm. 1.19] for $A(\mathbb{R}_+)$ with respect to the vacuum (= PCT transformation [20]), and $A_+(O) = A(I) \vee j(A(I))$.

We choose a system Δ of inequivalent irreducible DHR endomorphisms ρ_s localized in I, thus $\bar{\rho}_s = j \circ \rho_s \circ j$ are conjugates of ρ_s localized in J. We have

4.3 Lemma: Every irreducible subsector of Θ_+ is equivalent to some $\sigma\bar{\tau}$ with $\sigma, \tau \in \Delta$.

The proof of this Lemma is exactly as the proof of [29, Lemma 31].

Next, we show an analog of [29, Theorem 9], which implies that $\Theta \simeq \Theta_2$ indeed coincides with the dual canonical endomorphism Θ_+ for $A_+(O) \subset B_+^{\mathrm{ind}}(O)$:

4.4 Proposition: The multiplicities of $[\sigma\bar{\tau}]$ in the dual canonical endomorphism Θ_+ for $A_+(O) \subset B_+^{\mathrm{ind}}(O)$ equal $Z_{[\sigma][\tau]} = \dim \mathrm{Hom}(\alpha_\tau^+, \alpha_\sigma^-)$,

where $\alpha^\pm_\rho$ are the α-induced extensions of $\rho \in \Delta$ to the chiral net B (see App. B).

Proof: By Def. B.1, $\alpha^\pm_\rho$ are endomorphisms of $B(L)$, and by Prop. B.3, the global intertwiners coincide with the local intertwiners, i.e., $t\alpha^+_\tau(b) = \alpha^-_\sigma(b)t$ holds for all $b \in B$ iff it holds for all $b \in B(L)$, and in this case t belongs to $B(L)$.

Now, for $\sigma, \tau \in \Delta$, consider the space $X_{\sigma\tau}$ of intertwiners $\psi \in B^{\rm ind}_+(O)$ satisfying

$$\psi a = \sigma\bar\tau(a)\psi \qquad (a \in A_+(O)). \tag{4.7}$$

Then for $\psi \in X_{\sigma\tau}$, the same equation (4.7) also holds with $a \in A(K)$ and $a \in A(L')$ because then $\sigma\bar\tau(a) = a$ and because $B^{\rm ind}_+(O)$ commutes with $A(K)$ and $A(L')$. By strong additivity of A, (4.7) holds in fact for all $a \in A$.

Consider on the other hand the space $\mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$ of intertwiners $t \in B$ satisfying

$$t\alpha^+_\tau(b) = \alpha^-_\sigma(b)t \qquad (b \in B) \tag{4.8}$$

whose dimension is $Z_{[\sigma][\tau]}$. We claim that the map

$$\varphi : t \mapsto \psi := tR_\tau \qquad \varphi^{-1} : \psi \mapsto t := \sigma(\bar R^*_\tau)\psi \tag{4.9}$$

is an isomorphism between $\mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$ and $X_{\sigma\tau}$, which proves the proposition. Here, $R_\tau : \mathrm{id} \to \tau\bar\tau$ are the standard intertwiners in $A(L)$ as in [20, 29], normalized such that $R^*_\tau R_\tau = d(\tau)$, and $\bar R_\tau = \kappa_\tau \cdot R_\tau$ such that by [20, Proof of Lemma 3.5] one has $\tau(\bar R^*)R = \bar R^*\bar\tau(R) = 1$. The latter normalization condition ensures that the maps (4.9) are mutually inverse. It remains to show that the image of $\mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$ belongs to $X_{\sigma\tau}$, and vice versa.

Let $t \in \mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$. Then $t \in B(I)$ by Prop. B.3, and $\psi := \varphi(t) = tR_\tau$ by definition belongs to $B(L)$ and satisfies (4.7) for all $a \in A$. The non-trivial part is to show that it also belongs to $B(K)'$. Because $\sigma\bar\tau$ acts trivially on $A(K)$, ψ commutes with $A(K)$ by (4.7). Because $A(K)$ and v generate $B(K)$, where (γ, v, w) is the Q-system for $A(K) \subset B(K)$, it suffices to show that ψ commutes with $v \in B(K)$. We compute (with Prop. B.4(i))

$$\psi v = tR_\tau v = t\alpha^+_\tau\alpha^+_{\bar\tau}(v)R_\tau v = t\alpha^+_\tau(v)R_\tau = \alpha^-_\sigma(v)tR_\tau = vtR_\tau = v\psi, \tag{4.10}$$

because α^+_τ and α^-_σ act trivially on $v \in B(K)$. Thus, $\psi \in B(K)'$ and $\varphi(\mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)) \subset X_{\sigma\tau}$.

Conversely, let $\psi \in X_{\sigma\tau}$. By definition, $t := \varphi^{-1}(\psi)$ belongs to $B(L)$ and satisfies

$$t\tau(a) = \sigma(a)t \qquad (a \in A(L)). \tag{4.11}$$

Thanks to Prop. B.3, it remains to show that t has the required intertwining property

$$t\alpha^+_\tau(v) = \alpha^-_\sigma(v)t. \tag{4.12}$$

Inserting the above definitions for $t = \varphi^{-1}(\psi)$ and for $\alpha^\pm_\rho(v)$, we have

$$t\alpha^+_\tau(v) = \sigma(\bar{R}^*_\tau)\psi\varepsilon(\theta,\tau)v = \sigma(\bar{R}^*_\tau)\sigma\bar{\tau}(\varepsilon(\theta,\tau))\psi v = \sigma(\varepsilon(\theta,\bar{\tau})^*)\sigma\theta(\bar{R}^*_\tau)\psi v, \tag{4.13}$$

where $\theta = \gamma\restriction_A$ is localized in K, and

$$\alpha^-_\sigma(v)t = \varepsilon(\sigma,\theta)^* v\sigma(\bar{R}^*_\tau)\psi = \varepsilon(\sigma,\theta)^*\theta\sigma(\bar{R}^*_\tau)v\psi = \sigma\theta(\bar{R}^*_\tau)\varepsilon(\sigma,\theta)^* v\psi. \tag{4.14}$$

In (4.13), the statistics operator $\varepsilon(\theta,\bar{\tau})$ is trivial because of the ordering $J < K$ of the localizations of the endomorphisms, and in (4.14) $\varepsilon(\sigma,\theta)$ is trivial because $K < I$. Because $\psi \in X_{\sigma\tau}$ belongs to $B(K)'$ and $v \in B(K)$, ψ commutes with v, hence (4.13) and (4.14) are equal. Thus (4.8) holds, and $\varphi^{-1}(X_{\sigma\tau}) \subset \mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$. This completes the proof of the proposition. Q.E.D.

Proof of the Theorem (continued): The dimensions $Z_{[\sigma][\tau]}$ in the Prop. 4.4 being the multiplicities of $[\sigma\bar{\tau}]$ in Θ, we conclude $\Theta_+ = \Theta \simeq \bigoplus Z_{[\sigma][\tau]}\sigma\bar{\tau}$. For each pair σ, $\tau \in \Delta$ such that $\sigma\bar{\tau} \prec \Theta_+$, we fix a basis of charged intertwiners

$$\psi_i := \varphi(t_i) = t_i R_{\tau_i} \tag{4.15}$$

where t_i are bases of the spaces $\mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$ orthonormal with respect to their inner products $\langle t, t'\rangle = (d(\sigma)d(\tau))^{-1} \cdot R^*_\tau t^* t' R_\tau$. As σ and τ vary over Δ, we thus obtain a maximal system of charged intertwiners ψ_i in $B^{\mathrm{ind}}_+(O)$, $i = 1 \dots \sum Z_{[\sigma][\tau]}$, normalized as

$$\psi^*_i\psi_j = d(\sigma)d(\tau)\cdot\delta_{ij}. \tag{4.16}$$

We claim that these form an algebra of charged intertwiners with endomorphisms[11] $\varrho_i = \sigma_i\bar{\tau}_i \prec \Theta_+$ and coefficients Γ^k_{ij} as in (4.6), i.e., those of the α-induction construction given more explicitly in (4.17) below. Let us recall how the latter were determined.

The α-induction construction [40] proceeds by the specification of a Q-system in $A(I)\otimes A(I)^{\mathrm{opp}}$, which under the isomorphism between $A(I)^{\mathrm{opp}}$ and $A(J)$ given by $a^{\mathrm{opp}} \mapsto j(a^*)$ turns into the Q-system (Θ_2, W_2, X_2) in $A(I) \otimes A(J)$, determining the extension $A(I) \otimes A(J) \subset B^\alpha_2(O)$ up to isomorphism.

[11] The index i thus labels the irreducible components of $\Theta_+ \in \mathrm{End}(A_+(O))$, which may be pairwise equivalent whenever $Z_{[\sigma_i][\tau_i]} > 1$.

Applying in turn the natural isomorphism $A(I) \otimes A(J) \simeq A(I) \vee A(J)$, we read off [40, Sect. 3] [12]

$$\Gamma^k_{ij} = d(\Theta)^{\frac{1}{2}} \sum_{ef} \zeta^k_{ij,ef} \cdot T_e \, j(T_f) \in \mathrm{Hom}(\varrho_k, \varrho_i \varrho_j) \tag{4.17}$$

where $\varrho_i = \sigma_i \bar{\tau}_i \prec \Theta$, T_e form orthonormal bases of $\mathrm{Hom}(\sigma_k, \sigma_i\sigma_j) \subset A(I)$, T_f form orthonormal bases of $\mathrm{Hom}(\tau_k, \tau_i\tau_j) \subset A(I)$ and consequently $j(T_f)$ form orthonormal bases of $\mathrm{Hom}(\bar{\tau}_k, \bar{\tau}_i\bar{\tau}_j) \subset A(J)$, and the numerical coefficients $d(\Theta)^{\frac{1}{2}} \cdot \zeta^k_{ij,ef}$ are the expansion coefficients of

$$t_i \; \alpha^+_{\tau_i}(t_j) \in \mathrm{Hom}(\alpha^+_{\tau_i\tau_j}, \alpha^-_{\sigma_i\sigma_j}) \tag{4.18}$$

into the basis $T_e \, t_k \, T^*_f$ of $\mathrm{Hom}(\alpha^+_{\tau_i\tau_j}, \alpha^-_{\sigma_i\sigma_j})$:

$$t_i \alpha^+_{\tau_i}(t_j) = d(\Theta)^{\frac{1}{2}} \sum_{k,ef} \zeta^k_{ij,ef} \cdot T_e \, t_k \, T^*_f. \tag{4.19}$$

We now compute

$$\psi_i\psi_j = t_i R_{\tau_i} \cdot t_j R_{\tau_j} = t_i \alpha^+_{\tau_j} \alpha^+_{\bar{\tau}_j}(t_j) \cdot R_{\tau_i} R_{\tau_j} = t_i \alpha^+_{\tau_j}(t_j) \cdot R_{\tau_i} R_{\tau_j}, \tag{4.20}$$

because $\alpha^+_{\bar{\tau}_j}$ acts trivially on $B(I)$, and insert (4.19) as well as [29, Eq. (15)]

$$R_{\tau_i} R_{\tau_j} = \sum_g T_g \, j(T_g) \cdot R_{\tau_k}, \tag{4.21}$$

$T_g \in \mathrm{Hom}(\tau_k, \tau_i\tau_j)$. This yields

$$\psi_i\psi_j = d(\Theta)^{\frac{1}{2}} \sum_{k,ef} \zeta^k_{ij,ef} \cdot T_e \, t_k \, j(T_f) \cdot R_{\tau_k} = \Gamma^k_{ij} \cdot \psi_k \tag{4.22}$$

because $t_k \in B(I)$ and $j(T_f) \in A(J)$ commute.

It remains to prove the second of the two defining relations (A.9) with Γ^0_{ji} determined by (4.17). We observe that by definition of Γ^0_{ji} only τ_j conjugate to τ_i and σ_j conjugate to σ_i contribute, and the sums over e and f involve only one term $T_e \in \mathrm{Hom}(\mathrm{id}, \sigma_j\sigma_i)$ and $T_f \in \mathrm{Hom}(\mathrm{id}, \tau_j\tau_i)$. If we prove (the first of) the identities

$$\begin{aligned} d(\Theta)^{\frac{1}{2}} \textstyle\sum_j \zeta^0_{ji,ef} \cdot t^*_j T_e &= d(\tau_j) \cdot \alpha^+_{\tau_j}(t_i) T_f, \\ d(\Theta)^{\frac{1}{2}} \textstyle\sum_j \zeta^0_{ji,ef} \cdot T^*_f t^*_j &= d(\sigma_j) \cdot T^*_e \alpha^-_{\sigma_j}(t_i). \end{aligned} \tag{4.23}$$

[12] adapting the notation of [40] to the present conventions.

then the claim reduces to the corresponding result obtained in [29, eqs. (11) and (12)]: namely we get (using local commutativity of $t_j \in B(J)$ with $j(T_f) \in A(I)$ as well as $j(T_f^*)R_{\tau_j} = \tau_j(j(T_f^*))R_{\tau_j} \in \mathrm{Hom}(\bar\tau_i, \tau_j)$ and the trivial action of $\alpha^+_{\bar\tau_i}$ on t_i due to Prop. B.2)

$$\begin{aligned} \textstyle\sum_j \Gamma^{0*}_{ji}\psi_j = d(\Theta)^{\frac12} \sum_j \overline{\zeta^0_{ji,ef}} \cdot T_e^* t_j j(T_f^*) R_{\tau_j} \stackrel{(4.23)}{=} \\ = d(\tau_j) \cdot T_f^* \alpha^+_{\tau_j}(t_i^*) j(T_f^*) R_{\tau_j} \stackrel{\text{Prop. B.4}}{=} d(\tau_j) \cdot T_f^* j(T_f^*) R_{\tau_j} \alpha^+_{\bar\tau_i}(t_i^*) = \\ \stackrel{[29]}{=} R^*_{\tau_i} \alpha^+_{\bar\tau_i}(t_i^*) \stackrel{\text{Prop. B.2}}{=} R^*_{\tau_i} t_i^* = \psi_i^*. \end{aligned} \tag{4.24}$$

To prove (4.23), we choose $T_g \in \mathrm{Hom}(\mathrm{id}, \tau_i\tau_j)$ such that $\tau_i(T_f^*)T_g = \mathbf{1}$, and consequently also $T_f^*\tau_j(T_g) = \mathbf{1}$. Let $\tilde t := [T_e^* \alpha^-_{\sigma_j}(t_i T_g)]^* \in \mathrm{Hom}(\tau_i, \sigma_i)$. Then

$$d(\Theta)^{\frac12} \zeta^0_{ji,ef} = T_e^* \alpha^-_{\sigma_j}(t_i) t_j T_f = T_f^* \tilde t^* t_j T_f = d(\tau_i)^{-1} R^*_{\tau_j} \tilde t^* t_j R_{\tau_j} = d(\sigma_j) \langle \tilde t, t_j \rangle, \tag{4.25}$$

hence $\sum_j d(\Theta)^{\frac12} \zeta^0_{ji,ef} \cdot t_j^* = d(\sigma_j)\tilde t^*$ because t_j form an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$. Inserting the definitions, we get

$$\begin{aligned} d(\Theta)^{\frac12} \textstyle\sum_j \zeta^0_{ji,ef} \cdot t_j^* T_e = d(\sigma_j) \cdot \tilde t^* T_e = d(\sigma_j) \cdot T_e^* \alpha^-_{\sigma_j}(t_i T_g) T_e = \\ = d(\tau_j) \cdot T_f^* \alpha^+_{\tau_j}(T_g t_i) T_f = d(\tau_j) \cdot T_f^* \tau_j(T_g) \alpha^+_{\tau_j}(t_i) T_f = d(\tau_j) \cdot \alpha^+_{\tau_j}(t_i) T_f, \end{aligned} \tag{4.26}$$

using the fact that T_e and T_f implement standard left-inverses for the α-induced sectors, and the trace property for standard left-inverses. Similarly,

$$\begin{aligned} d(\Theta)^{\frac12} \textstyle\sum_j \zeta^0_{ji,ef} \cdot T_f^* t_j^* = d(\sigma_j) \cdot T_f^* \tilde t^* = d(\sigma_j) \cdot T_f^* T_e^* \alpha^-_{\sigma_j}(t_i T_g) = \\ = d(\sigma_j) \cdot T_e^* \alpha^-_{\sigma_j}(t_i \tau_i(T_f^*) T_g) = d(\sigma_j) \cdot T_e^* \alpha^-_{\sigma_j}(t_i), \end{aligned} \tag{4.27}$$

proving (4.23).

This completes the proof of Theorem 4.1. Q.E.D.

Proof of Corollary 4.2: Obvious.

Remark: In 2D CFT, there is a pair of maximal left and right chiral algebras such that $A_L(I) \otimes A_R(J) \subset A_L^{\max}(I) \otimes A_R^{\max}(J) \subset B_2(O)$. Under standard assumptions [39], these are given by $A_L^{\max}(I) = B_2(I \times J) \cap (\mathbf{1} \otimes A_R(J))'$ (independent of J) and similar for $A_R^{\max}$. In the present situation, with $A_L = A_R = A$ and $B_2 = B_2^\alpha$, the isomorphism of Thm. 4.1 identifies $A_L^{\max}(I)$ with $B(L) \cap B(L \setminus \bar I)$. Namely, $B_+^{\mathrm{ind}}(O) \cap A(J)' = B(L) \cap (B(K) \vee A(J))'$ and $B(K) \vee A(J) = \{v^K\} \vee A(K) \vee A(J) = \{v^K\} \vee A(L \setminus \bar I) = B(L \setminus \bar I)$ by strong additivity of A. In particular, the intersection $B(L) \cap B(L \setminus \bar I)$ does

not depend on the upper boundary of the interval L and may be replaced by $\hat{A}_L(I) := B((a,\infty)) \cap B((b,\infty))'$ if $I = (a,b)$.

The chiral nets $I \mapsto \hat{A}_L(I)$ and $I \mapsto \hat{A}_R(I) := B((-\infty,b)) \cap B((-\infty,a))'$ thus define two local and mutually local chiral nets, both extending $I \mapsto A(I)$ within $B(I)$, such that $A(I) \vee A(J) \subset \hat{A}_L(I) \vee \hat{A}_R(J) \subset B_+^{\mathrm{ind}}(O)$ for $J < I$. In the setting of [4], they correspond to the intermediate subfactors $N \subset M_\pm \subset M$.

5 Bi-localized charge structure in BCFT

Our aim in this section is to establish in the algebraic framework formulae of the type (1.11), (1.12), exhibiting a separation of the left and right charges of local fields in BCFT (*bi-localized charge structure*). This will explain Cardy's observation [11] concerning the relation between n-point local correlation functions and $2n$-point conformal blocks in a model-independent setting. Furthermore, it enables us to compute the specific linear coefficients which guarantee locality, in terms of the DHR structure of the underlying net A of chiral observables.

5.1. Preliminaries.

Let us recall and adapt for our present purposes several results from the literature.

In [12], under the name of *field bundle* a "crossed product action of the DHR category on the observables" has been constructed as a first substitute for an algebra of charged fields has been constructed. The fibres of this bundle were labelled by all the DHR endomorphisms. The huge redundancy has been eliminated with the "reduced field bundle" in [15, 16] where only one fibre was retained for each irreducible superselection sector.

This amounts to a choice, for each irreducible sector $[s]$, of a representative DHR endomorphism ρ_s along with the representation of the observables on the Hilbert space $\mathcal{H}_s$. As a *space*, H_s coincides with the vacuum Hilbert space $\mathcal{H}_0$ of the net A, but as a *representation* it differs in that A is represented on $\mathcal{H}_s$ under the action of the endomorphism ρ_s, i.e., $\pi_s = \rho_s$. We call $\hat{\mathcal{H}}$ the direct sum of the $\mathcal{H}_s$ (which is finite because A is rational), and $\hat{\pi}$ the corresponding representation.

Let σ be a DHR endomorphism of A and T_e an orthonormal basis of intertwiners $T_e : \rho_s \to \rho_t\sigma$, $e = 1 \dots \dim \mathrm{Hom}(\rho_s, \rho_t\sigma)$. Then T_e, as an

operator from $\mathcal{H}_s$ to $\mathcal{H}_t$, satisfies the intertwining relation

$$T_e \pi_s(a) = \pi_t(\sigma(a)) T_e. \tag{5.1}$$

It is crucial that T_e, although an element of A as an operator, must not be considered as an *observable* since it acts on $\mathcal{H}_s$ in the representation $\pi_0 = \mathrm{id}$, and not in the representation π_s pertaining to $\mathcal{H}_s$. We emphasize this fact by our notation, and denote by ψ_e^σ the operator on $\hat{\mathcal{H}}$ which coincides with T_e on the subspace $\mathcal{H}_s$ (with values in $\mathcal{H}_t$) and is extended by zero on its orthogonal complement in $\hat{\mathcal{H}}$.[13] Thus

$$\psi_e^\sigma \hat{\pi}(a) = \hat{\pi}(\sigma(a)) \psi_e^\sigma. \tag{5.2}$$

If σ is localized in an interval I, then $\sigma(a) = a$ for $a \in A(I')$, hence ψ_e^σ commutes with $\hat{\pi}(A(I'))$. We therefore arrive at the "reduced field net" [15, 16] of von Neumann algebras $F_{\mathrm{red}}(I)$, which are generated by $\hat{\pi}(A(I))$ and the charged intertwiners ψ_e^σ with σ localized in I. This net is relatively local with respect to the subnet $\hat{\pi}(A)$, but non-local itself. The reduced field net is covariant w.r.t. the unitary representation $\mathcal{U}_{\hat{\pi}}$ implementing covariance of the observables. The operators ψ_e^σ satisfy braid-group commutation relations with numerical coefficients ("R-matrices") determined by the DHR statistics operators $\varepsilon(\sigma_1, \sigma_2)$. They are bounded operator versions of the chiral exchange fields discussed in the Introduction.

If u is a charge transporter $u : \sigma \to \hat{\sigma}$ with $\hat{\sigma}$ localized in $\hat{I}$, then

$$\hat{\pi}(u) \psi_e^\sigma = \psi_e^{\hat{\sigma}} \tag{5.3}$$

belongs to $F_{\mathrm{red}}(\hat{I})$. As discussed in [16], suitable regularized limits of ψ_e^σ as the localization of σ shrinks to a point y, behave like point-like chiral "exchange fields", generalizing $a(y)$, $b(y)$ and their adjoints displayed in (1.10). Since the commutation relations survive in the limit, the latter satisfy commutation relations with the same R-matrices as the former. Their correlations converge to (primary or descendant, depending on the details of the limit chosen) conformal blocks, whose analytical monodromy properties thus represent the R-matrices of the DHR statistics.

The reduced field net does not comply with the axioms for a (non-local) chiral extension of A (in the sense of [33] or [1]) because its local algebras are not factors (and as a consequence, the vacuum vector in $\mathcal{H}_0$ is a cyclic, but not a separating vector). However, every (non-local) field extension B on a Hilbert space $\mathcal{H}_B$ can be "embedded" in (an amplification of) the

[13] The same operator was denoted $F_e(1)^*$ in [16].

reduced field net as follows [33]. Let (γ, v, w) be the Q-system associated with the inclusion $\pi(A) \subset B$, and (θ, w, x) be the dual Q-system. θ is a DHR endomorphism of A; we may write [14]

$$[\theta] = \bigoplus_{s:\text{inequivalent irreducibles}} n^s[s] = \bigoplus_{p:\text{irreducibles}} [s(p)]. \tag{5.4}$$

Consequently $x : \theta \to \theta^2$ has an expansion of the form

$$x = \sum_{p,q,r;e} \lambda^r_{pq}(e) \cdot w^p \rho_p(w^q)\; T_e\; w^{r*} \tag{5.5}$$

where $\rho_p \equiv \rho_{s(p)}$ are the representatives of the sectors $[s(p)]$ as before, $w^p : \rho_p \to \theta$ form a complete system of orthonormal isometries in A, and $T_e \in A$ form orthonormal bases of $\mathrm{Hom}(\rho_r, \rho_p\rho_q)$. The numerical coefficients $\lambda^r_{pq}(e) \in \mathbb{C}$ are "generalized Clebsch-Gordon coefficients" characteristic for the inclusion $\pi(A(I)) \subset B(I)$. Then, the charged isometry $v \in B$ can be represented in terms of operators from $F_{\rm red}$ as

$$v = \sum_{p,q,r;e} \lambda^r_{pq}(e) \cdot E^p \hat{\pi}(w^q)\; \psi^{\rho_q}_e\; E^{r*} = \sum_{p,q,r;e} \lambda^r_{pq}(e) \cdot \pi(w^q) E^p\; \psi^{\rho_q}_e\; E^{r*} \tag{5.6}$$

where $E^p : \hat{\mathcal{H}} \to \mathcal{H}_B$ are the partial isometries which identify the irreducible subrepresentation $\mathcal{H}_{s(p)} \subset \hat{\mathcal{H}}$ with the irreducible subrepresentation $\mathcal{H}_p \subset \mathcal{H}_B$, and are zero on the complement. It follows that the charged intertwiners $\psi_q := d(\theta)^{\frac{1}{2}} \cdot \pi(w^{q*})v$ of the chiral extension B (cf. App. A) arise as the characteristic linear combinations

$$\psi_q = d(\theta)^{\frac{1}{2}} \sum_{p,r;e} \lambda^r_{pq}(e) \cdot E^p\; \psi^{\rho_q}_e\; E^{r*} \tag{5.7}$$

of charged intertwiners from $F_{\rm red}$ (possibly amplified by multiplicities of sectors $[s]$ in $\mathcal{H}_B$). The algebras $B(I)$ generated by these linear combinations do have the vacuum as a cyclic *and* separating vector. Remarkably, in case B is local (or graded local), then the specific linear combinations (5.7) satisfy (graded) local commutativity, although the individual summands ψ^ρ_e also in this case satisfy proper braid group commutation relations.

5.2. Application to BCFT.

[14] In the sequel, indices $s, t, \dots$ label the irreducible DHR sectors, while $p, q, \dots$ label the irreducible subrepresentations of π which may come with multiplicities: $\pi \simeq \bigoplus \pi_p = \bigoplus n^s \pi_s(p)$. Indices $i, j, \dots$ will label the irreducible components of Θ_+ as in Sect. 4.

After these preliminaries, we return to boundary CFT. We formulate the main result of this section:

5.1 Proposition: Let σ, $\bar{\tau}$ be irreducible DHR endomorphisms, localized in I and J, respectively, such that $\sigma\bar{\tau} \prec \Theta_+$. Then the charged intertwiners $\psi_i \in B_+^{\rm ind}(O)$, $i = 1 \dots Z_{[\sigma][\tau]}$, for the inclusion $\pi(A_+(O)) \subset B_+^{\rm ind}(O)$ can be represented as

$$\psi_i = \sum_{p,q;g,h} \varphi^p_{q,i}(g,h) \cdot E^q \; \psi^\sigma_g \; \psi^{\bar{\tau}}_h \; E^{p*} \tag{5.8}$$

with numerical coefficients $\varphi^p_{q,i}(g,h)$ to be specified in Cor. 5.2 below. Here, the sums over p and q extend over the irreducible subrepresentations of π, h and g stand for orthonormal bases of intertwiners $T_h : \rho_{s(p)} \to \rho_t\bar{\tau}$ and $T_g : \rho_t \to \rho_{s(q)}\sigma$, respectively, and sum over the intermediate sectors $[t]$ is implicit in the sum over the "channels" g and h.

Proof: Let us first consider the case of the reference double-cone $O = I \times J = (y,z) \times (-z,-y)$ as discussed in Sect. 4. We recall from (4.15) that the operators $\psi_i = \varphi(t_i) = t_i\pi(R_\tau) \in B_+^{\rm ind}(O) \subset B$ are intertwiners $\psi_i : {\rm id}_B \to \alpha^-_\sigma \alpha^+_{\bar{\tau}}$. Equivalently (because A and v generate B), they satisfy

$$\psi_i\pi(a) = \pi(\sigma\bar{\tau}(a))\psi_i \qquad (a \in A) \tag{5.9}$$

and

$$\psi_i v = \alpha^-_\sigma \alpha^+_{\bar{\tau}}(v)\psi_i = \pi[\sigma(\varepsilon(\theta,\bar{\tau}))\varepsilon(\sigma,\theta)^*]v\psi_i. \tag{5.10}$$

Now let $\mathcal{U} : \pi \to \theta$ implement the unitary equivalence between the representation π of A on $\mathcal{H}_B$ and the representation through the DHR endomorphism θ on $\mathcal{H}_0$, and let $\varphi_i := {\rm Ad}_\mathcal{U}(\psi_i)$. Under ${\rm Ad}_\mathcal{U}$, (5.9) and (5.10) translate into

$$\varphi_i\theta(a) = \theta\sigma\bar{\tau}(a)\varphi_i, \tag{5.11}$$

i.e., $\varphi_i \in {\rm Hom}(\theta, \theta\sigma\bar{\tau}) \subset A$, and in addition the linear condition on φ_i

$$\varphi_i \, x = \theta[\sigma(\varepsilon(\theta,\bar{\tau}))\varepsilon(\sigma,\theta)^*]x \, \varphi_i, \tag{5.12}$$

because ${\rm Ad}_\mathcal{U}(v) = x$ [33]. Finally, the normalization (4.16) of ψ_i turns into the normalization

$$\varphi_i^*\varphi_j = d(\sigma)d(\tau) \cdot \delta_{ij}. \tag{5.13}$$

Introducing a basis of the space ${\rm Hom}(\theta, \theta\sigma\bar{\tau})$, we conclude:

5.2 Corollary: Consider the finite linear problem (5.12) to be solved within the DHR category of A, i.e., $\varphi_i \in {\rm Hom}(\theta, \theta\sigma\bar{\tau})$. Let φ_i be its solutions, subject to the normalization (5.13). They have an expansion

$$\varphi_i = \sum_{p,q;g,h} \varphi^p_{q,i}(g,h) \cdot w^q T_g T_h w^{p*} \tag{5.14}$$

where $w^p = \mathcal{U}^* E^p \upharpoonright_{\mathcal{H}_{s(p)}} : \rho_p \to \theta$ are orthonormal isometries. Transformed back to $\mathcal{H}_B$, we have (5.8), where the numerical coefficients $\varphi^p_{q,i}(g,h)$ are given by

$$\varphi^p_{q,i}(g,h) = T^*_h T^*_g w^{q*} \varphi_i w^p \in \mathrm{Hom}(\rho_p, \rho_p) = \mathbb{C}. \tag{5.15}$$

This concludes the proof of Prop. 5.1 in the case of the reference double-cone O. Now, we may change the localization to any other double-cone $\hat{O} = \hat{I} \times \hat{J}$. Similar as in (5.3), we multiply ψ_i from the left with the charge transporter $\pi(U_\sigma \sigma(U_{\bar\tau}))$ where $U_\sigma : \sigma \to \hat\sigma$ and $U_{\bar\tau} : \bar\tau \to \hat{\bar\tau}$ with the desired localizations. From $\pi(\sigma(U_{\bar\tau}))E^q \psi^\sigma_g = E^q \hat\pi(\sigma(U_{\bar\tau}))\psi^\sigma_g = E^q \psi^\sigma_g \hat\pi(U_{\bar\tau})$ and (5.3), we conclude that (5.8) in fact holds for the charged intertwiners associated with arbitrary double-cones $\hat{O}$, substituting only $\hat\sigma$ for σ and $\hat{\bar\tau}$ for $\bar\tau$. Dropping the $\hat{}$ symbols, we may equally well assert that the structure (5.8) holds for any double-cone.

This completes the proof of Prop. 5.1 in the general case. Q.E.D.

We note that eq. (5.12) is quite similar to the condition Def. 5.5 in [19].

Note that ψ^σ_g belongs to $F_{\mathrm{red}}(I)$, and $\psi^{\bar\tau}_h$ belongs to $F_{\mathrm{red}}(J)$. We have thus geometrically separated the "left" and "right" charges of the charged intertwiners, by representing them as linear combinations of *bilocalized products* of charged intertwiners from $F_{\mathrm{red}}(I)$ and $F_{\mathrm{red}}(I)$. The specific coefficients $\varphi^p_{q,i}(g,h)$, arising through the solution of a linear problem in the DHR category involving the dual canonical endomorphism θ (Cor. 5.2), are algebraic invariants for the (non-local) chiral extension $\pi(A) \subset B$.

Assuming the same regularity of the point-like limits $\hat{O} \to (t,x)$ as in [16], we infer the convergence of n-point correlations of $\psi_i(t,x)$ to characteristic linear combinations of $2n$-point conformal blocks involving the arguments $t+x$ and $t-x$. (Clearly, the limit cannot be effectuated by the action of the Möbius group. Instead, one has to use the local implementers which implement the local action of the Möbius group on $F_{\mathrm{red}}(I)$ and act trivially on $F_{\mathrm{red}}(J)$, and vice versa, to obtain a local action of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ on $F_{\mathrm{red}}(I) \vee F_{\mathrm{red}}(J)$ and hence on $B(O)$.)

The coefficients $\varphi^a_{b,i}(g,h)$ are affected neither by the transport from O to $\hat{O}$ nor (up to some overall normalization) by the point-like limit. We conclude that Cardy's observation, originally derived from Ward identities in minimal models, is in fact a model-independent feature of boundary CFT, reflecting purely algebraic structures of the associated (non-local) chiral extension $\pi(A) \subset B$. The relative coefficients of the representation of local n-point correlation functions as linear combinations of $2n$-point conformal blocks are the products of n coefficients $\varphi^p_{q,i}(g,h)$ according to the contribut-

ing channels.

It should be remarked that, according to the structure (5.8), while the initial and final sectors of ψ_i necessarily belong to $\mathcal{H}_B$, the intermediate sectors $[t]$ may range over all DHR sectors of the chiral net A, as can be nicely seen in the examples (1.11) and (1.12). The correlation functions of boundary CFT therefore carry information also about those chiral sectors which are not present in the Hilbert space of its local fields.

6 Varying the boundary conditions

As we have seen, the chiral extension $I \mapsto B(I)$ of $I \mapsto A(I)$ determines not only the Hilbert space $\mathcal{H}_B \simeq \bigoplus_s n^s \mathcal{H}_s$ of the boundary CFT, but also the detailed charge structure of its local fields as in (5.8) and, as a consequence illustrated by the example (1.8), (1.9), the behavior of the local fields and their correlations close to the boundary $x = 0$.

In this section, we want to *vary the boundary conditions* by varying the (non-local) chiral extension B. As is well known from [6], there is a finite system of inequivalent (non-local) chiral extensions B_a which all give rise to the same coupling matrix $Z_{[\sigma][\tau]}$. In the language of modular categories [17, 34, 30], these extensions correspond to Morita equivalent Frobenius algebras [19, 38]. We want to show here, that they even give rise to boundary CFT's with locally isomorphic subfactors $A_+(O) \subset B^{\rm ind}_{a,+}(O)$. In view of Thm. 4.1, this means that they all share the local structure of the same Minkowski space CFT B_2^α.

Our result is essentially a corollary to a result in [5] making use of [6].

Let $\iota : A(I) \to B(I)$ be the inclusion homomorphism for a given (non-local) chiral extension $\pi(A) \subset B$, and consider the system $\mathcal{X} = \{a : A(I) \to B(I)\}$ of inequivalent irreducible subhomomorphisms of $\iota \circ \rho$ as ρ ranges over the DHR endomorphisms of A localized in I.[15]

Each $a \in \mathcal{X}$ naturally gives rise to a Q-system (θ_a, w_a, x_a) (where $\theta_a =$

[15] In the terminology of categories [18, 30, 37] (where a Q-system is a Frobenius algebra), $a \in \mathcal{X}$ are the irreducible modules of the Frobenius algebra, cf. e.g., [18, Lemma 5.24 and Chap. 6], forming the objects of the module category. If B is local, the Frobenius algebra is commutative. In this case, Kirillov and Ostrik have shown [30] that the module category is again a monoidal (tensor) category. In the same situation, Böckenhauer and Evans have found [3, Thm. 3.9] a bijection between the elements $a \prec \iota\rho$ of $\mathcal{X}$ and the irreducible subendomorphisms $\beta \prec \alpha^\pm_\rho$ of α-induced endomorphisms ($\dim \mathrm{Hom}(\iota\rho, \iota\sigma) = \dim \mathrm{Hom}(\alpha^\pm_\rho, \alpha^\pm_\sigma)$). These two results relate to each other in such a way that the monoidal product $a_1 \times a_2$ coincides with the composition of endomorphisms $\beta_1 \circ \beta_2$. In the non-local case, there is no such bijection.)

$\bar{a}a \prec \bar{\rho}\,\theta\rho$ is a DHR endomorphism because $a \prec \iota\rho$) and hence defines an inclusion $A \subset B_a$ as in [13, 5] ("varying the iota vertex"), i.e., (non-local) chiral extensions $\pi_a(A) \subset B_a$ with inclusion homomorphisms ι_a such that $\theta_a = \bar{\iota}_a\iota_a$. We may call the family B_a (as a varies over $\mathcal{X}$) the *DHR orbit* of the given extension B. (Warning: The association $a \mapsto B_a$ is in general not injective, see below.)

Each member of the DHR orbit induces a boundary CFT $B^{\mathrm{ind}}_{a,+}$ as well as a Minkowski space CFT $B^{\alpha}_{a,2}$ by the α-induction construction. Although the associated representations $\pi_a \simeq \theta_a$ in general differ from each other, the following local isomorphy holds.

6.1 Proposition: The inclusions $A_+(O) \subset B^{\mathrm{ind}}_{a,+}(O)$ are isomorphic throughout the DHR orbit. The same holds true for the inclusions $A_2(O) \subset B^{\alpha}_{a,2}(O)$.

Proof: We recall from Sect. 4, that the algebraic structure of the subfactors of interest is coded in the numerical coefficients $\zeta^k_{ij,ef}$ of their Q-systems. The latter, in turn, arise as expansion coefficients (4.19) of the monoidal product $t_i\alpha^+_{\tau_i}(t_j)$ of intertwiners $t : \alpha^+_\tau \to \alpha^-_\sigma$ between α-induced endomorphisms of both signs. In [5, p. 21], a bijection β_a between the intertwiner spaces $\mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma)$ and $\mathrm{Hom}(\alpha^+_{a,\tau}, \alpha^-_{a,\sigma})$ for α-inductions to the extensions B_a within a DHR orbit was established. It is therefore sufficient to show that this bijection respects the monoidal product.

Let $M = B(I)$ and $N = A(I)$. For $a \in \mathcal{X}$, let $\bar{a} : M \to N$ be a conjugate homomorphism, and $\bar{a}(M) \subset N \subset M_a$ the Jones basic construction [26] associated with the subfactor $\bar{a}(M) \subset N$. Let $\iota_a : N \to M_a$ be the inclusion homomorphism of N into M_a, and $\bar{\iota}_a : M_a \to N$ a conjugate homomorphism such that $\bar{\iota}_a(M_a) = \bar{a}(M) \subset N$. Then $\varphi_a = \bar{a}^{-1} \circ \bar{\iota}_a : M_a \to M$ is an isomorphism.

Now, if ρ and $\bar{\rho}$ are conjugate DHR endomorphisms of A localized in I and $a \prec \iota\rho|_N$, then $\theta_a = \bar{\iota}_a\iota_a = \bar{a}a$ is contained in (the restriction to N of) $\bar{\rho}\bar{\iota}\iota\rho = \bar{\rho}\,\theta^I\rho$ which is again a DHR endomorphism localized in I. The statistics operators $\varepsilon^\pm(\tau, \theta_a)$ enter the definition of α-induction $\alpha^\pm_{a,\tau}$, cf. App. B. According to [6, p. 455f], if $T \in \mathrm{Hom}(a, \iota\rho) \subset M$ is isometric, then

$$U^\pm_\tau = T^*\iota(\varepsilon^\pm(\tau,\rho))\alpha^\pm_\tau(T) \in \mathrm{Hom}(\alpha^\pm_\tau a, a\tau) \subset M \tag{6.1}$$

is unitary, and $\varepsilon^\pm(\tau, \theta_a) = \bar{a}(U^\pm_\tau)\varepsilon^\pm(\tau, \bar{a}\iota)$. One finds

$$\alpha^\pm_{a,\tau} = \varphi_a^{-1} \circ \mathrm{Ad}_{U^\pm_\tau} \circ \alpha^\pm_\tau \circ \varphi. \tag{6.2}$$

Consequently, the bijection $\beta_a : \mathrm{Hom}(\alpha^+_\tau, \alpha^-_\sigma) \to \mathrm{Hom}(\alpha^+_{a,\tau}\alpha^-_{a,\sigma})$ is given by

$$\beta_a(t) = \varphi_a^{-1}(U^-_\sigma t U^{+*}_\tau). \tag{6.3}$$

In order to show that β_a respects the monoidal product, we have to show that

$$\varphi_a^{-1}\big(U^-_{\sigma_1\sigma_2}t_1\cdot\alpha^+_{\tau_1}(t_2)U^+_{\tau_1\tau_2}{}^*\big)=\varphi_a^{-1}\big(U^-_{\sigma_1}t_1U^{+*}_{\tau_1}\big)\cdot\alpha^+_{a,\tau_1}\varphi_a^{-1}\big(U^-_{\sigma_2}t_2U^{+*}_{\tau_2}\big) \quad (6.4)$$

which due to (6.2) is equivalent to

$$U^-_{\sigma_1\sigma_2}t_1\cdot\alpha^+_{\tau_1}(t_2)U^+_{\tau_1\tau_2}{}^*=U^-_{\sigma_1}t_1U^{+*}_{\tau_1}\cdot U^+_{\tau_1}\alpha^+_{\tau_1}(U^-_{\sigma_2}t_2U^{+*}_{\tau_2})U^{+*}_{\tau_1}. \quad (6.5)$$

Using "naturality" of the DHR braiding with respect to α-induction, as expressed, e.g., in [6, Eq. (14)], we find

$$U^+_{\tau_1}\alpha^+_{\tau_1}(U^+_{\tau_2})=U^+_{\tau_1\tau_2} \quad (6.6)$$

and similar for U^-_σ, which implies (6.5). This completes the proof. Q.E.D.

Since the structure of the local subfactors in the case of Minkowski space extensions B_2 of $A\otimes A$ determines the global structure (thanks to the "unbroken symmetry", i.e., existence of a global conditional expectation in this case, cf. Sect. 2), the associated two-dimensional theories $B^\alpha_{a,2}$ may in fact be considered as identical.

In contrast, the boundary CFT nets $B^{\rm ind}_{a,+}$ are defined on different Hilbert spaces $\mathcal{H}_a$ given by $\pi_a\simeq\theta_a=\bar\iota_a\iota_a$. In particular, in spite of the algebraic isomorphism of the local subfactors, the corresponding bi-localized charge structures as in Prop. 5.1 differ among different members within the DHR orbit. As a consequence, exemplified by the example (1.11) and (1.12), also the scaling behavior of the local fields towards the boundary differs.

The DHR orbit associates several BCFT's to a given one. E.g., the "Cardy case" discussed in the literature [11, 14, 46] is the DHR orbit of the trivial extension $B=A$, $\iota=\mathrm{id}$, which includes $B_+=A_+^{\rm dual}$. The elements of $\mathcal{X}$ in this case are labelled by the sectors $\iota_\rho\equiv\rho$ of A. To be more specific, the Hilbert spaces $\mathcal{H}_\rho$ carry the representation $\pi_\rho\simeq\theta_\rho\equiv\bar\rho\rho$ of A and hence of A_+ and $A_+^{\rm dual}$. Thus, in the Cardy case, the members of the DHR family are just the extensions $\pi_\rho(A_+)\simeq\bar\rho\rho(A_+)\subset\bar\rho\rho(A_+^{\rm dual})$. The non-trivial charge structure of the "charged fields" of $A_+^{\rm dual}$ arises through the non-trivial action of $\bar\rho\rho$ on the charge transporters $u:\sigma^I\to\sigma^J$ (cf. Remark 3 after Def. 2.1). In the Ising model, there are three sectors $[0]$, $[\frac{1}{16}]$, and $[\frac12]$. The corresponding chiral extensions B_s are, in turn, A itself, CAR (cf. Sect. 2), and again A itself (exemplifying the non-injectivity of the association $a\mapsto B_a$). The boundary field nets $B^{\rm ind}_{s,+}$ are generated by A_+ and, in turn, charged intertwiners of the structure ϕ_0 as in (1.11), ϕ_1 as in (1.12), and again ϕ_0. In fact, $B^{\rm ind}_{0,+}$ and $B^{\rm ind}_{\frac12,+}$ both coincide with $A_+^{\rm dual}$. A

more refined structure distinguishing between 0 and $\frac{1}{2}$ will be discussed in the next section.

The main problem, however, is the classification of the other orbits, if there are any. By the results of Sect. 2, this amounts to the classification of non-local chiral extensions $\pi(A) \subset B$, reformulated according to App. A as the classification of Q-systems in the DHR category of superselection sectors of A.[16] From As explained in Sect. 3.2, this is a finite-dimensional problem and it has only finitely many solutions. Of course, complete classifications can be expected only when the chiral observables A are specified, see e.g., [28].

We speculate that each DHR orbit of non-local chiral extensions $A \subset B_a$ contains a distinguished element which is local, at least if the coupling matrix $Z_{[\sigma][\tau]}$ is of type I [6]. The argument could go like this. Every element of the DHR orbit defines the same theory B_2 on Minkowski space by the α-induction prescription, see above. This theory in turn has a pair of maximal chiral subalgebras $A_L^{\max} \supset \pi_L^{\max}(A)$ and $A_R^{\max} \supset \pi_R^{\max}(A)$, where $\pi_L^{\max}$ and $\pi_R^{\max}$ are determined by the "vacuum block" of the coupling matrix $Z_{[\sigma][\tau]}$. (We expect that these coincide with $\hat{A}_L$ and $\hat{A}_R$ mentioned in the end of Sect. 4.) If Z is of type 1, $\pi_L^{\max}$ and $\pi_R^{\max}$ are equivalent, and we may suppress the subscript. We conjecture that the local chiral extension $\pi^{\max}(A) \subset A^{\max}$ is a distinguished *local* element of the orbit. Thus, classification of DHR orbits of BCFT would be reduced to classification of local chiral extensions, cf. [28, 33], or of commutative Frobenius algebras [30]. We hope to return to this conjecture in a separate work.

7 Partition functions and modularity

We mention in this section aspects of modular invariant partition functions, as far as they can be easily derived in our framework. Let us recall, however (cf. Sect. 1), that in our approach Modular Invariance of the partition function is not a first principle. Therefore, the natural appearance of the matrix Z (both as the coupling matrix of left and right chiral sectors in the Minkowski space theory B_2^α and as the coupling matrix for the bi-localized charge structure, cf. Prop. 4.4), its automatic modular invariance [6], and the validity of relations (7.4) and (7.7) below also in the flat space QFT framework, is a remarkable fact about the intrinsic structure of Minkowski space CFT with or without boundary.

[16] See also [38, Thm. 1] according to which every module category arises as the module category of some Frobenius algebra.

The structure of the system $\mathcal{X} = \{a \prec \iota\rho \text{ irreducible}\}$ associated with a BCFT (cf. Sect. 6) defines a "nimrep" (non-negative integer matrix representation) of the fusion rules $[s][t] = \bigoplus_u N_u^{st}[u]$ of the superselection sectors. Namely, if a belongs to $\mathcal{X}$, then $a \prec \iota\rho$ for some DHR endomorphism ρ, hence then $a\rho_s \prec \iota\rho\rho_s$, and every irreducible component of $a\rho_t$ again belongs to $\mathcal{X}$. Hence

$$[a\rho_s] = \bigoplus_{b\in\mathcal{X}} n_{ab}^s[b] \qquad \text{with} \qquad \sum_b n_{ab}^s n_{bc}^t = \sum_u N_u^{st} n_{ac}^u. \tag{7.1}$$

This implies that the diagonal matrix elements n_{aa}^s are the multiplicities of ρ^s within $\theta_a = \bar{a}a$, thus

$$\mathcal{H}_a = \mathcal{H}_{B_a} = \bigoplus_s n_{aa}^s \mathcal{H}_s \equiv \bigoplus_s n_{aa}^s \mathcal{H}_s. \tag{7.2}$$

In the literature on boundary CFT in Statistical Mechanics (for a review see, e.g., [46]) one discusses also theories defined on Hilbert spaces

$$\mathcal{H}_{ab} = \bigoplus_s n_{ab}^s \mathcal{H}_s. \tag{7.3}$$

This leads us to consider "non-diagonal" boundary CFT nets $O \mapsto B_{ab,+(O)} \supset \pi_{ab}(A_+(O))$ and the associated (non-local) chiral nets $I \mapsto B_{ab}(I) \supset \pi_{ab}(A(I))$ which are defined on $\mathcal{H}_{ab}$ carrying the DHR representation $\pi_{ab} \simeq \bar{a}b$, for any pair $a, b \in \mathcal{X}$. These theories arise through the reducible subfactor associated with $\theta = (\bar{a} \oplus \bar{b}) \circ (a \oplus b)$. If $a \neq b$, the Hilbert spaces H_{ab} do not contain the vacuum vector (because $n_{ab}^0 = \delta_{ab}$), so that the standard theory of chiral extensions as applied in Sect. 2–6 cannot be used. We expect nevertheless (without elaborating) that the results of the previous sections largely carry over to these theories as well, and allow to make precise contact with the Statistical Mechanics interpretation along the following lines.

The partition function for the spectrum of the chiral conformal Hamiltonian of the boundary CFT on $\mathcal{H}_{ab} = \bigoplus_s n_{ab}^s \mathcal{H}_s$ is

$$Z_{ab}(\beta) = \mathrm{Tr}_{\mathcal{H}_{ab}} \pi_{ab}(\exp -\beta(L_0 - \tfrac{c}{24})) = \sum_s n_{ab}^s \, \chi_s(\beta). \tag{7.4}$$

n^s being a nimrep of the (commutative) fusion rules, its joint spectrum is given by the matrix elements of the modular matrix S (note that in the algebraic approach, complete rationality implies non-degeneracy of the braiding [29], and hence the DHR statistics defines a unitary representation of the modular group $SL(2,\mathbb{Z})$ [16]), i.e., one has the "Cardy equation"

$$n_{ab}^s = \sum_t \psi_{at} \frac{S_{st}}{S_{0t}} \psi_{bt}^*. \tag{7.5}$$

Inserting this expansion in the partition function Z_{ab}, and taking for granted the modular transformation law of the chiral characters $\chi_s(\beta)$, one obtains

$$Z_{ab}(\beta) = \sum_t \psi^*_{bt}\chi_t(\hat{\beta})\psi_{at}, \tag{7.6}$$

where $\hat{\beta} = 4\pi^2/\beta$ is the modular transform of the inverse temperature β. Usually [46], the right-hand side of this formula is reinterpreted as a matrix element of the conformal Hamiltonian of the Minkowski space theory between a pair of so-called "Ishibashi boundary states" $|a\rangle = \sum_t \psi_{at}|t\rangle$, which weakly realize the boundary condition $T_L = T_R$:

$$Z_{ab}(\beta) = \langle b| \exp -\tfrac{1}{2}\hat{\beta}(L_{L0} + L_{R0} - \tfrac{c}{12})|a\rangle. \tag{7.7}$$

These Ishibashi states, however, are linear combinations of non-normalizable vectors in the Hilbert spaces $\mathcal{H}_t \otimes \mathcal{H}_t$. It was pointed out, e.g., in [22] that Ishibashi states, rather than vector states on $A \otimes A$, should be considered as KMS (= Gibbs in this case) states on A, where the second copy of A appears via Tomita's Modular Theory [43, Chap. VI, Thm. 1.19] as the commutant of A in the GNS representation of the KMS state.

While we have not elaborated these issues, we hope to arrive, in a future publication, at a better algebraic understanding of the structures outlined in this section.

8 Conclusion

We have classified boundary conformal quantum field theories in terms of chiral extensions of the underlying local chiral observables A. These extensions, which are in general non-local, are in turn classified in terms of Q-systems (Frobenius algebras) in the DHR modular category of superselection sectors of A. We have analysed how general structural properties of the chiral observables are transmitted to the local algebras of the BCFT. Among other things, we have shown the absence of DHR superselection sectors of the latter (Sect. 2).

A chiral extension determines both a BCFT and a Minkowski space CFT. Well away from the boundary, these two theories are algebraically indistinguishable (Sect. 4). Only near the boundary, the breakdown of symmetry changes the algebraic structure. This effect is exhibited in the bi-localized charge structure of the local fields in the BCFT. This structure can be derived (and explicitly computed) from the superselection structure (the DHR

modular category) of the chiral observables. It may be regarded as an algebraic invariant for the embedding of the latter into the full theory (Sect. 5). The bi-localized charge structure in turn determines the scaling behavior of the local fields with $x \to 0$. In this sense, the boundary "conditions" on the non-chiral fields of BCFT are in fact rather a derived feature.

BCFT's associated with the same chiral observables can be grouped into families ("DHR orbits") which are algebraically isomorphic well away from the boundary, but differ near the boundary. The members of each orbit may thus be interpreted as the different ways a Minkowski space CFT may "react" to the presence of a boundary; but it can (in general) not be considered as different representations of the same abstract theory on the half-space (Sect. 6).

Each DHR orbit is accompanied by a "nimrep" of the fusion rules of the chiral observables, which controls the modular behavior of the partition function of the conformal Hamiltonian (Sect. 7).

One may also study boundary CFT with two boundaries [46], corresponding to a QFT on a strip $0 < x < L$. The formulae (1.2), (1.4), (1.16) etc. for the chiral observables pertain to that situation as well, provided $t \pm x$ are interpreted as angular coordinates of the circle, adjusted with a normalization factor $L/2\pi$, rather than cartesian coordinates of the lightlike axes. We refrain in this article from elaboration on local extensions of the chiral observables on the strip.

A Q-systems and algebras of charged intertwiners

We give a brief reminder of the notion of *Q-system* associated with a subfactor $N \subset M$ of type *III* von Neumann algebras, and then present a Lemma concerning the generation of M in terms of charged intertwiners. This lemma is the obvious generalization of an argument used in [29] in a special case.

A subfactor $N \subset M$ is irreducible if $N' \cap M = \mathbb{C} \cdot \mathbf{1}$. The *index* $[M : N]$ is the optimal bound $\lambda \geq 1$ such that there is a conditional expectation $\mathcal{E} : M \to N$ satisfying the lower operator bound $\mathcal{E}(m^*m) \geq \lambda^{-1} \cdot m^*m$. The *dimension* $d(\rho)$ of an endomorphism $\rho \in \mathrm{End}(N)$ is the square root of the index $[N : \rho(N)]$.

The condition of finite index is equivalent to the property that, with $\iota : N \to M$ the inclusion homomorphism, there is a "conjugate" homomorphism $\bar\iota : M \to N$ and a "canonical" pair of isometric intertwiners $v : \mathrm{id}_M \to \gamma := \iota\bar\iota \in \mathrm{End}(M)$ in M and $w : \mathrm{id}_N \to \theta := \bar\iota\iota \in \mathrm{End}(N)$ in N, such that

$\iota(w)^*v = \lambda^{-\frac{1}{2}}\mathbf{1}_M$ and $\bar\iota(v)^*w = \lambda^{-\frac{1}{2}}\mathbf{1}_N$. Then, $\mathcal{E}(m) = \iota(w)^*\gamma(m)\iota(w)$ is the (unique, if $N \subset M$ is irreducible) conditional expectation. γ and θ are the "canonical" and "dual canonical" endomorphisms associated with the subfactor, and $d(\gamma) = d(\theta) = \lambda = [M : N]$.

A *Q-system* in M is a triple (ρ, T, S) where $\rho \in \mathrm{End}(M)$ is an endomorphism of M, and T and S are isometric intertwiners $T : \mathrm{id} \to \rho$ and $S : \rho \to \rho^2$ in M, satisfying the relations [17]

$$T^*S = \rho(T^*)S = \lambda^{-\frac{1}{2}} \cdot \mathbf{1}, \qquad SS = \rho(S)S \tag{A.1}$$

A Q-system in M determines a subfactor $N \subset M$ of index λ in terms of data of M as the image $N := \mathcal{E}(M)$ of the conditional expectation $\mathcal{E} : M \to N$, defined by $\mathcal{E}(m) := T^*\rho(m)T$. Thus, the Q-system for $N \subset M$ is $(\gamma, v, \iota(w))$. Likewise, the Q-system in N for $\bar\iota(M) \subset N$ (the *dual Q-system for* $N \subset M$) is $(\theta, w, \bar\iota(v))$.

By Jones' "basic construction" [26], a subfactor $N \subset M$ determines (up to unitary equivalence) another subfactor $M \subset M_1$, isomorphic to $\bar\iota(M) \subset N$. The basic construction applied to $\bar\iota(M) \subset N$, recovers $N \subset M$ (up to isomorphism), hence $N \subset M$ is also determined by its *dual* Q-system (θ, w, x) in N.

Given a Q-system (θ, w, x) in N, a *concrete* realization of M results if one finds a representation of N in a Hilbert space $\mathcal{H}$ and an isometry v in $\mathcal{B}(\mathcal{H})$ such that (γ, v, w) form a Q-system in $M := N \vee \{v\}$ where γ extends θ by setting $\gamma(v) := x$. Then, with ι the inclusion map of N into M via its representation on $\mathcal{H}$, (γ, v, w) is the Q-system for $N \subset M$, and (θ, w, x) is the dual Q-system. We make use of this constructive scheme in Sect. 4.

The conditions on v which ensure that (γ, v, w) form a Q-system with $\gamma\restriction_N = \theta$ and $\gamma(v) = x$, can be formulated as an *algebra of charged intertwiners*, as follows.

Let $N \subset M$ be a subfactor of finite index, and (γ, v, w) and (θ, w, x) its Q-system and dual Q-system, and

$$\theta(n) = \sum_i w^i \varrho_i(n) w^{i*} \qquad (n \in N) \tag{A.2}$$

the decomposition of θ into irreducibles (choosing representatives $\rho_i = \rho_j$ whenever ρ_i and ρ_j are equivalent), $\varrho_0 = \mathrm{id}$, $w^0 = w$. Then the *charged intertwiners*

$$\psi_i := d(\theta)^{\frac{1}{2}} \cdot w^{i*}v \in M \tag{A.3}$$

[17] In more general frameworks, such as Frobenius algebras in tensor categories [18], one has to require in addition an equivalent of $SS^* = \rho(S^*)S$; in a C* context as ours, the latter relation follows from the remaining ones [34, Sect. 6].

satisfy

$$\psi_i n = \varrho_i(n)\psi_i \qquad (n \in N); \tag{A.4}$$

we say that "ψ_i carry charge ϱ_i". The charged intertwiner for $\varrho_0 = \mathrm{id}$ is

$$\psi_0 = \mathbf{1}, \tag{A.5}$$

and whenever $\varrho_i = \varrho_j$, one has the normalization

$$\psi_i^* \psi_j = d(\varrho_i) \cdot \delta_{ij}. \tag{A.6}$$

Together with N, the charged intertwiners generate M; more precisely, every element of M has a unique expansion

$$m = \sum_k n_k \psi_k, \qquad (n_k = \mathcal{E}(m\psi_k^*) \in N). \tag{A.7}$$

As $x \in N$ is an intertwiner $x : \theta \to \theta^2$, it has a unique expansion

$$x = d(\theta)^{-\frac{1}{2}} \sum_{ijk} w^i \varrho_i(w^j)\, \Gamma_{ij}^k\, w^{k*} \tag{A.8}$$

with intertwiners $\Gamma_{ij}^k : \varrho_k \to \varrho_i\varrho_j$ in N. Transscribing the relations of the Q-system $vv = \gamma(v)v = xv$ and $v^* = d(\theta)^{\frac{1}{2}} w^* vv^* = d(\theta)^{\frac{1}{2}} w^*\gamma(v^*)v = d(\theta)^{\frac{1}{2}} w^* x^* v$ in terms of the charged intertwiners, one arrives at

$$\psi_i\psi_j = \sum_k \Gamma_{ij}^k\, \psi_k, \qquad \psi_i^* = \sum_j \Gamma_{ji}^{0*}\, \psi_j. \tag{A.9}$$

We note that, by (A.6) alone, $\sum_i t_i\psi_i = 0$ with $N \ni t_i : \varrho_i \to \varrho$ implies $t_i = 0$. Indeed, multiplying the sum from the left by $\psi_j^* s^*$ ($N \ni s : \varrho_j \to \rho$) one gets $s^* t_j = 0$. Since s is arbitrary, $t_j = 0$. Therefore, the relations (A.4–6) and (A.9) among the charged intertwiners impose constraints on the "coefficients" Γ_{ij}^k, e.g., $\Gamma_{0j}^k = \delta_{jk}\mathbf{1}$, $\Gamma_{i0}^k = \delta_{ik}\mathbf{1}$, as well as $\sum_n \Gamma_{ij}^n \Gamma_{nk}^l = \sum_m \varrho_i(\Gamma_{jk}^m)\Gamma_{im}^l$ (associativity of the expansion (A.9)). Furthermore, $\Gamma_{ij}^0 : \mathrm{id} \to \varrho_j\varrho_i$ vanish unless ϱ_j is conjugate to ϱ_i, and $\sum_k \Gamma_{ij}^{k*}\Gamma_{ij}^k d(\varrho_k) = d(\varrho_i)d(\varrho_j)$.

The relation

$$\sum_{ij} \Gamma_{ij}^{k*}\Gamma_{ij}^{k'} = d(\theta)\delta_{kk'} \cdot \mathbf{1} \tag{A.10}$$

is of a different status: it follows from $x^*x = 1$, but seemingly not from the relations among the charged intertwiners alone.

This set of relations (A.4–6) and (A.9) in M is, like the Q-system (γ, v, w), a complete invariant for the subfactor $N \subset M$, see Lemma A.2.

A.1 Definition: Let ϱ_i be a finite system of pairwise either inequivalent or equal irreducible endomorphisms of N of finite dimension among which $\varrho_0 = \mathrm{id}$ occurs precisely once, and $\Gamma^k_{ij} \in \mathrm{Hom}(\varrho_k, \varrho_i\varrho_j) \subset N$. An *algebra of charged intertwiners* for N is a system of operators ψ_i satisfying (a) the intertwining property (A.4) for ϱ_i, (b) the normalizations as in (A.5), (A.6), and (c) the algebra (A.9) with coefficients satisfying (A.10), where $d(\theta) := \sum_i d(\varrho_i)$.

The first statement of the following lemma summarizes the above discussion; the second statement is the converse: the algebra of charged intertwiners determines the subfactor and its Q-system. While a special case underlies the argument leading to Cor. 45 of [29], we think it appropriate to formulate the general case.

A.2 Lemma: (i) An irreducible subfactor $N \subset M$ determines an algebra of charged intertwiners. In particular, the condition (A.10) on the coefficients is automatic, if they arise in this way.

(ii) Let $\psi_i \in \mathcal{B}(\mathcal{H})$ be an algebra of charged intertwiners for N with endomorphisms $\varrho_i \in \mathrm{End}(N)$ and coefficients $\Gamma^k_{ij} \in N$, and M be the algebra generated by N and ψ_i. Then the subfactor $N \subset M$ has Q-system (γ, v, w) and dual Q-system (θ, w, x), where (in turn) θ is defined as in (A.2) with the help of any complete orthogonal system of isometries $w^i \in A$, $w := w^0$, x is defined as in (A.8), $v := d(\theta)^{-\frac{1}{2}} \sum_i w^i\psi_i$, and by definition γ extends θ by $\gamma(nv) := \theta(n)x$.

(iii) Two algebras of charged intertwiners with the same endomorphisms in $\mathrm{End}(N)$ and coefficients in N give rise to isomorphic subfactors (possibly on different Hilbert spaces), with the isomorphism given by identification of the charged intertwiners.

Sketch of the proof: (ii) It is straightforward to see that the relations of the algebra of charged intertwiners ensure the following: $wn = \theta(n)w$, $x\theta(n) = \theta^2(n)x$, $vn = \theta(n)v$ $(n \in N)$; $w^*w = \mathbf{1}$, $v^*v = \mathbf{1}$, $x^*x = \mathbf{1}$; $\theta(w^*)x = d(\theta)^{-\frac{1}{2}}\mathbf{1}$, $w^*x = w^*\gamma(v) = d(\theta)^{-\frac{1}{2}}\mathbf{1}$, $w^*v = d(\theta)^{-\frac{1}{2}}\mathbf{1}$; $vv = xv$, $xx = \theta(x)x$; and $v^* = d(\theta)^{\frac{1}{2}} \cdot w^*x^*v$. These include the defining relations for (θ, w, x) to be a Q-system in N. The missing information for (γ, v, w) to be a Q-system is that γ is an endomorphism. It is also straightforward to see that γ respects products and the previous relations, and

$$\gamma(v^*) = d(\theta)^{\frac{1}{2}} \cdot \theta(w^*x^*)x = d(\theta)^{\frac{1}{2}} \cdot \theta(w^*)xx^* = x^* = \gamma(v)^*. \qquad \text{(A.11)}$$

Because $\psi_i = d(\theta)^{\frac{1}{2}} \cdot w^{i*}v$ by definition of v, v and N generate M, so γ

is indeed an endomorphism. Hence (γ, v, w) is a Q-system, and (θ, w, x) its dual ($\theta = \gamma \restriction_N$ and $x = \gamma(v)$). Finally $N = w^*\gamma(M)w$ because $w^*\gamma(vnv^*)w = d(\theta)^{-1} \cdot n$, showing that the Q-systems are indeed the Q-system associated with $N \subset M$ and its dual. The subfactor is irreducible since id is contained in θ with multiplicity one.

(iii) is now obvious. Q.E.D.

B α-induction

We collect a number of well-known results on α-induction [33, 3], used in the course of our arguments in this article.

B.1 Definition [33]: Let N be a factor, and Δ a set of endomorphisms ϱ of N equipped with a braiding $\varepsilon(\varrho_1, \varrho_2) : \varrho_1\varrho_2 \to \varrho_2\varrho_1$, giving rise to a braided C* tensor category with direct sums and subobjects. Let $N \subset M$ be an irreducible subfactor with canonical endomorphism γ, and dual canonical endomorphism $\theta = \gamma \restriction_N$ such that $\theta \in \Delta$. Then for $\varrho \in \Delta$,

$$\alpha^+_\varrho := \gamma^{-1} \circ \mathrm{Ad}_{\varepsilon(\theta,\varrho)} \circ \varrho \circ \gamma \tag{B.1}$$

extends the endomorphism $\varrho \in \Delta$ of N to an endomorphism α^+_ϱ of M. (The nontrivial fact is that $\mathrm{Ad}_{\varepsilon(\theta,\varrho)} \circ \varrho \circ \gamma(M)$ belongs to $\gamma(M)$.) One has

$$\alpha^+_\varrho(n) = \varrho(n) \qquad \text{and} \qquad \alpha^\pm_\varrho(v) = \varepsilon(\theta, \varrho)v \tag{B.2}$$

for $n \in N$ and v the canonical isometry in the Q-system (γ, v, w). The same holds true for α^-_ϱ, replacing the braiding in (B.1), (B.2) with the opposite braiding $\varepsilon^-(\varrho_1, \varrho_2) = \varepsilon(\varrho_2, \varrho_1)^*$. The endomorphisms $\alpha^\pm_\varrho$ are invariant under inner conjugations of the Q-system (i.e., $\gamma \mapsto \mathrm{Ad}_u \circ \gamma$, $v \mapsto uv$, $w \mapsto uw$, $u \in N$ unitary).

B.2 Proposition [33]: If $I \mapsto [A(I) \subset B(I)]$ is a (chiral) quantum field theoretical net of subfactors, then the dual canonical endomorphism of $A(I)$ for $A(I) \subset B(I)$ extends to a DHR endomorphism θ of the net A with $[\theta]$ independent of I. With Δ the set of DHR endomorphisms and $\varepsilon(\rho_1, \rho_2)$ the DHR braiding [15], $\alpha^\pm_\rho$ can be defined as endomorphisms of the net B such that (B.2) and its analog for α^-_ρ still hold. If ρ is localized in I, then α^+_ρ (resp. α^-_ρ) is a semi-localized endomorphism of B, i.e., it acts trivially on $B(K)$ whenever $I < K$ (resp. $I > K$).

B.3 Proposition: Let ρ, σ be localized in I. Then for every combination of $\varepsilon = \pm$, $\varepsilon' = \pm$, the space of global intertwiners

$$\{t \in B : t\alpha^\varepsilon_\rho(b) = \alpha^{\varepsilon'}_\sigma(b)t \text{ for all } b \in B\} \tag{B.3}$$

coincides with the space of local intertwiners

$$\{t \in B(I) : t\alpha^{\varepsilon}_{\rho}(b) = \alpha^{\varepsilon'}_{\sigma}(b)t \text{ for all } b \in B(I)\}. \tag{B.4}$$

Proof: Every element $b \in B$ can be written uniquely as $b = av$ with $a \in A$ and $v \in B(I)$ the charged intertwiner of the Q-system (γ, v, w) for $A(I) \subset B(I)$ [33]; furthermore $b \in B(I)$ iff $a \in A(I)$. Let $t = av$ be a global intertwiner. Then $t\alpha^{\varepsilon}_{\rho}(a_1) = \alpha^{\varepsilon'}_{\sigma}(a_1)t$ implies that a is a global intertwiner in $\mathrm{Hom}(\theta\rho, \sigma)$, hence $a \in A(I)$ by Haag duality of A, hence $t \in B(I)$ is in fact a local intertwiner.

Conversely, let $t = av$ be a local intertwiner. Then a is local intertwiner, hence [20, Thm. 2.3] a global intertwiner in $\mathrm{Hom}(\theta\rho, \sigma)$. Thus, we have $t\alpha^{\varepsilon}_{\rho}(b) = \alpha^{\varepsilon'}_{\sigma}(b)t$ for all $b \in B(I)$ by assumption, and for all $b \in A(I')$ by the trivial action of the endomorphisms on $A(I')$ and locality. Since $B(I)$ and $A(I')$ generate all of B by strong additivity of A, t is a global intertwiner. Q.E.D.

B.4 Proposition [3, Lemma 3.5 and Lemma 3.25]: Let ρ, σ, τ be DHR endomorphisms of A.

(i) If $T \in \mathrm{Hom}(\rho, \sigma) \subset A$, then also $T \in \mathrm{Hom}(\alpha^{\pm}_{\rho}, \alpha^{\pm}_{\sigma}) \subset B$.

(ii) If $t \in \mathrm{Hom}(\alpha^{\pm}_{\rho}, \alpha^{\pm}_{\sigma})$, then the naturality relations $\alpha^{\pm}_{\tau}(t)\varepsilon(\rho, \tau) = t\varepsilon(\sigma, \tau)$ and $\alpha^{\pm}_{\tau}(t)\varepsilon(\tau, \rho)^* = t\varepsilon(\tau, \sigma)^*$ hold.

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References

[1] C. D'Antoni, R. Longo, F. Radulescu: *Conformal nets, maximal temperature and models from free probability*, J. Oper. Theory **45** (2001) 195–208 [e-print math.OA/9810003].

[2] C. D'Antoni, K. Fredenhagen, S. Köster: *Implementation of conformal covariance by diffeomorphism symmetry*, Lett. Math. Phys. **67** (2004) 239–247.

[3] J. Böckenhauer, D.E. Evans: *Modular invariants, graphs and α-induction for nets of subfactors*, Commun. Math. Phys. **197** (1998) 361–386.

[4] J. Böckenhauer, D.E. Evans: *Modular invariants from subfactors: type I coupling matrices and intermediate subfactors*, Commun. Math. Phys. **213** (2000) 267–289.

[5] J. Böckenhauer, D.E. Evans: *Modular invariants and subfactors*, Fields Inst. Commun. **30** (2001) 11–37 [e-print math.OA/0008056].

[6] J. Böckenhauer, D.E. Evans, Y. Kawahigashi: *On α-induction, chiral generator and modular invariants for subfactors*, Commun. Math. Phys. **208** (1999) 429–487

[7] J. Böckenhauer, D.E. Evans, Y. Kawahigashi: *Longo-Rehren subfactors arising from α-induction*, Publ. RIMS (Kyoto) **37** (2001) 1–35 [e-print math.OA/0002154].

[8] H.-J. Borchers: *Energy and momentum as observables in quantum field theory*, Commun. Math. Phys. **2** (1966) 49–54.

[9] D. Buchholz, C. D'Antoni, R. Longo: *Nuclear maps and modular structures II: application to quantum field theory*, Commun. Math. Phys. **129** (1990) 115–138.

[10] R.E. Behrend, P.A. Pearce, V.C. Petkova, J.-B. Zuber: *Boundary conditions in rational conformal field theories*, Nucl. Phys. **B 579** (2000) 707–773.

[11] J. Cardy: *Conformal invariance and surface critical behavior*, Nucl. Phys. **B 240** (1984) 514–532.

[12] S. Doplicher, R. Haag, J.E. Roberts: *Local observables and particle statistics, 1+2*, Commun. Math. Phys. **23** (1971) 199-230, Commun. Math. Phys. **35** (1974) 49–85

[13] D.E. Evans: *Fusion rules of modular invariants*, Rev. Math. Phys. **14** (2002) 709–732.

[14] G. Felder, J. Fröhlich, J. Fuchs, C. Schweigert: *Correlation functions and boundary conditions in RCFT and three-dimensional topology*, Compos. Math. **131** (2002) 189–237 [e-print hep-th/9912239].

[15] K. Fredenhagen, K.-H. Rehren, B. Schroer: *Superselection sectors with braid group statistics, 1*, Commun. Math. Phys. **125** (1989) 201–226.

[16] K. Fredenhagen, K.-H. Rehren, B. Schroer: *Superselection sectors with braid group statistics, II*, Rev. Math. Phys. **SI1** (Special issue, 1992) 113–157.

[17] J. Fuchs, C. Schweigert: *Solitonic sectors, α-induction and symmetry breaking boundaries*, Phys. Lett. **B 490** (2000) 163–172.

[18] J. Fuchs, C. Schweigert: *Category theory for conformal boundary conditions*, Fields Inst. Commun. **39** (2003) 25–71, [e-print math.CT/0106050].

[19] J. Fuchs, I. Runkel, C. Schweigert: *TFT construction of RCFT correlators: partition functions*, Nucl. Phys. **B 646** (2002) 353–497.

[20] D. Guido, R. Longo: *The conformal spin and statistics theorem*, Commun. Math. Phys. **181** (1996) 11–35.

[21] R. Haag: Local Quantum Physics, Springer Verlag, Berlin – Heidelberg – New York, 1996

[22] K.C. Hannabuss, M. Semplice: *Boundary conformal fields and Tomita-Takesaki theory*, J. Math. Phys. **44** (2003) 5517–5529.

[23] M. Izumi, H. Kosaki: *On a subfactor analogue of the second cohomology*, Rev. Math. Phys. **14** (2002) 733–757.

[24] M. Izumi, R. Longo, S. Popa: *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras*, J. Funct. Anal. **155** (1998) 25–63.

[25] R. Jaffe: *Unnatural acts: unphysical consequences of imposing boundary conditions on quantum fields*, AIP Conf. Proc. **687** (2003) 3–12 [e-print hep-th/0307014].

[26] V. Jones: *Index for subfactors*, Invent. Math. **72** (1983) 1–25.

[27] Y. Kawahigashi: *Generalized Longo-Rehren subfactors and α-induction*, Commun. Math. Phys. **226** (2002) 269–287.

[28] Y. Kawahigashi, R. Longo: *Classification of local conformal nets: case $c < 1$*, to appear in Ann. Math. [e-print math-ph/0201015].

[29] Y. Kawahigashi, R. Longo, M. Müger: *Multi-interval subfactors and modularity of representations in conformal field theory*, Commun. Math. Phys. **219** (2001) 631–669.

[30] A. Kirillov, V. Ostrik: *On a q-analog of the McKay correspondence and the ADE classification of $\widehat{sl}_2$ conformal field theories*, Adv. Math. **171** (2002) 183–227 [e-print math.QA/0101219].

[31] R. Longo: *A duality for Hopf algebras and for subfactors, I*, Commun. Math. Phys. **159** (1994) 133–150.

[32] R. Longo: *Conformal subnets and intermediate subfactors*, Commun. Math. Phys. **237** (2003) 7–30.

[33] R. Longo, K.-H. Rehren: *Nets of subfactors*, Rev. Math. Phys. **7** (1995) 567–597

[34] R. Longo, J.E. Roberts: *A theory of dimension*, K-Theory **11** (1997) 103–159.

[35] R. Longo, F. Xu: *Topological sectors and dichotomy in conformal field theory*, to appear in Commun. Math. Phys. [e-print math.OA/0309366].

[36] M. Müger: *Superselection structure of massive quantum field theories in 1+1 dimensions*, Rev. Math. Phys. **10** (1998) 1147–1170.

[37] M. Müger: *From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Algebra **180** (2003) 81–157.

[38] V. Ostrik: *Module categories, weak Hopf algebras and modular invariants*, e-print math.QA/0111139.

[39] K.-H. Rehren: *Chiral observables and modular invariants*, Commun. Math. Phys. **208** (2000) 689–712.

[40] K.-H. Rehren: *Canonical tensor product subfactors*, Commun. Math. Phys. **211** (2000) 395–406.

[41] K.-H. Rehren: *Locality and modular invariance in 2D conformal QFT*, Fields Inst. Commun. **30** (2001) 341–354 [e-print math-ph/0009004].

[42] K.-H. Rehren, B. Schroer: *Exchange algebra on the light-cone and order/disorder $2n$-point functions in the Ising field theory*, Phys. Lett. **B 198** (1987) 84–88.

[43] M. Takesaki: Theory of Operator Algebras II, Springer Encyclopedia of Mathematical Sciences, Vol. **125** (2003).

[44] V.G. Turaev: *Modular categories and 3-manifold invariants*, Int. J. Mod. Phys. **B6** (1992) 1807–1824.

[45] F. Xu: *On a conjecture of Kac-Wakimoto*, Publ. RIMS (Kyoto) **37** (2001) 165–190 [e-print math.RT/9904098].

[46] J.-B. Zuber: *CFT, BCFT, ADE and all that*, e-print hep-th/0006151.